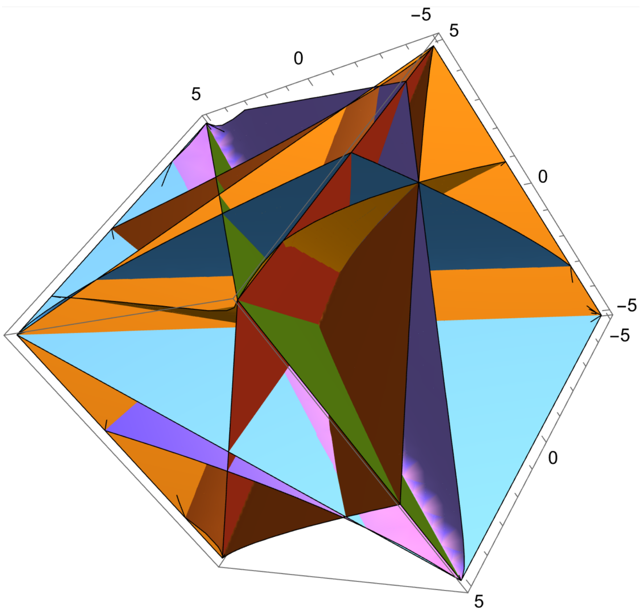
UNIVERSITY OF COPENHAGEN
DEPARTMENT OF MATHEMATICAL SCIENCES



Master's thesis

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Polar Varieties and Applications

Date: May 2026

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ABSTRACT. We introduce the notion of classical polar varieties and present equivalent ways to define them. Our initial definition is given in [10], then we give two additional approaches to defining them and two computational approaches to computing them, which were initially defined in [5, 32]. Afterwards, we introduce the notion of polarity using a nondegenerate quadric to define reciprocal polar varieties as given in [25, 28], and demonstrate how the affine reciprocal polar varieties are related to the critical points of the Euclidean distance problem given in [17]. We introduce the higher-order osculating spaces given in [31] to define higher-order polar varieties and higher-order reciprocal polar varieties used in [29]. In the preceding chapter, we demonstrate the relationship between the degrees of classical polar varieties called polar degrees and distance optimization questions. Finally, we define higher-order polyhedral distance loci using higher-order normal space and introduce an ideal that contains it.

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Date:	31 05 2026

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1 Introduction

The main objects of this thesis are polar varieties $P(X, V)$, their rational classes $p_i(X)$ called polar classes and their degrees $\mu_i(X)$ are called polar degrees. Polar varieties $P(X, V)$ are subvarieties of projective varieties $X \subseteq \mathbb{P}^n$ that records non-transversal intersections relative to projective subspace V. They were initially defined to compute characteristic classes of smooth manifolds, and to classify the projective varieties. A modern treatment of polar varieties was developed in the 1930s by Francesco Severi (1879-1961) in [33, 34] and afterwards by John Arthur Todd (1908-1994) in [37, 36].

The history of polarity and reciprocity, dates back even further to the works of Blaise Pascal (1623-1662) on conics, and Jean-Victor Poncelet (1788-1867) on poles associated with conic section, leading to the principle of duality in projective space. These notions led to the development of the earlier notions of reciprocal polar varieties. For more historical discussion on polar varieties, we refer to the [27] and [35].

With the development of modern algebraic geometry, polar classes became a tool to compute invariants such as Chern classes and Segre classes using polar classes [7, 13, 14, 20]. Beyond their geometric applications, they also have applications in applied algebraic geometry; to name a few: solving real polynomial systems [3, 4, 5, 6], the calculation of witness sets [25, 32], optimization questions such as the Euclidean distance degree [17], polyhedral norm optimization [40], the calculation of bottle neck degree [30], and critical points [16]. This list is not exhaustive, but shows their importance in applied mathematics.

This thesis investigates the geometry of classical polar varieties, reciprocal polar varieties, and higher-order polar varieties, with a focus on their applications to distance optimization problems in metric algebraic geometry.

We start Section 2 with preliminaries to build a projective geometry and linear algebra background that we use throughout the thesis. Afterwards, the rest of Section 2 is devoted to understanding classical polar varieties, their geometry in projective and affine spaces. Then, we present two equivalent definitions, one of which allows us to see them as a pullback of special Schubert varieties through the Gauss map $\gamma_X^{-1}(\Sigma_m)$, and the other as a projection of a pullback of dual linear subspace $\pi_1(\pi_2^{-1}(V^\vee))$, where the notion of dual is defined through conormal variety $CN(X)$; both have different advantages over one another. In the last part of Section 2, we give two different computational approaches to defining polar varieties using rank conditions on matrices, relating the classical polar varieties to papers [32] and [2].

In Section 3, we first introduce the notion of polarity $\perp$ in projective space $\mathbb{P}^n$ using a non-singular quadric $Q = \mathbb{V}(q)$, which allows us to work with orthogonal complements in projective space. Afterwards, by taking the polar of projective subspace $K_i^\perp$, we define the reciprocal polar varieties $W^\perp_{K_i}(X)$ as a subvariety of projective variety X, by the nontransversal intersection of X with $(K_i^\perp + p)^\perp$. We give many examples in the affine case, and investigate the behavior of the affine reciprocal polar varieties when the flag of projective subspaces $\mathcal{K}$ is contained within the hyperplane at infinity H_∞. Finally, we prove how the degree of affine reciprocal polar varieties is the Euclidean distance degree which is defined to be number of complex critical points of Euclidean distance optimization.

The main goal of Section 4 is to first define osculating spaces $\mathbb{T}_p^k(f)$ which are generalizations of projective tangent space $T_p(f)$, and generalize classical and reciprocal polar varieties to higher-order classical and reciprocal polar varieties. In the first part of Section 4, we change our language from defining equations $\mathbb{V}(F_1, \ldots, F_s)$ to morphisms to projective space $f : X \hookrightarrow \mathbb{P}^n$, in order to define osculating spaces $\mathbb{T}_p^k(f)$ using jet matrices $A_p^{(k)}(f)$. We compute the general formula of osculating spaces for degree 1,2 and 3 curves in $\mathbb{A}^3$. In the end we define higher-order classical polar variety $P_{k,i}(f, V)$, and define the higher-order normal spaces $\mathbb{N}_p^{(k)}(f, Q)$ to define the higher-order reciprocal polar variety $W_k^\perp(f, K_i)$.

In Section 5, we begin with an introduction to intersection theory to build the necessary language to work with polar degrees $\mu_i(X)$, and prove that polar degree does not depend on the projective subspace V for generic V. Afterwards, we prove the relationship between polar degrees $\mu_i(X)$ and multidegrees $\delta_j(X)$, where multidegrees $\delta_j(X)$ are defined to be coefficients of rational class of conormal variety $[CN(X)]$. Then using the results of [24], we prove many interesting properties of polar degrees $\mu_i(X)$. As a first application, we use the reciprocal polar varieties $W_{K_i}^\perp(X)$ to prove a well known result stating that Euclidean distance degree $\mathrm{EDdegree}(X)$ equals to the sum of polar degrees $\sum_{i=0}^m \mu_i(X)$. As a second application, we first define the polyhedral distance which is a norm $\|\cdot\|_B$ whose unit ball B is a centrally symmetric full dimensional polytope. Then we discuss the polyhedral distance optimization problem, which is an optimization problem where norm is given by polyhedral distance. An example for polyhedral distance optimization is optimal transport in algebraic statistics given by polyhedral norm Wasserstein distance. In the end of the chapter, we show how polar degrees $\mu_i(X)$ governs the complexity of the polyhedral distance optimization problem.

After studying, the existing notions in the literature, we use the last Section 6 to show future directions. In particular, we look at higher-order polyhedral distance optimization problem, using the similar approach given in [31] for Euclidean distance optimization. For a curve given as a image of a morphism $f : \mathbb{A}^1 \to \mathbb{A}^n$ with full rank $(n-1)$-jet matrix, we define higher-order distance loci $\mathcal{PL}_{n-1}(f)$, using the $(n-1)$-th order higher-order normal spaces $\mathbb{N}_p^{(n-1)}(f)$. We then give an ideal description of this contruction and prove that it contains the higher-order distance loci $\mathcal{PL}_{n-1}(f)$.

2 Classical Polar Variety

2.1 Preliminaries

Since we are defining most of our objects in projective space $\mathbb{P}^n$, we begin by recalling some constructions and facts that will be used throughout.

Let $\mathbf{k}$ be a field of characteristic zero, in most of the cases, we work over complex numbers $\mathbf{k} = \mathbb{C}$ or else real numbers $\mathbf{k} = \mathbb{R}$. The projective space $\mathbb{P}^n_{\mathbf{k}}$ is the set of all 1-dimensional lines in $\mathbb{A}^{n+1}_{\mathbf{k}}$. If the ground field $\mathbf{k}$ is clear from the context, we denote the projective space as $\mathbb{P}^n$ and affine space as $\mathbb{A}^n$ without the subscripts.

The notation for the affine cone of $\mathbb{P}^n$ is always $\mathbb{A}^{n+1}$. Recall that we say a subset $\hat{V} \subset \mathbb{A}^{n+1}$ is an affine cone if $0 \in \hat{V}$ and $\hat{V}$ is closed under scalar multiplication, that is, for all $\lambda \in \mathbf{k}$ and $\hat{p} \in \hat{V}$ we have $\lambda\hat{p} \in \hat{V}$. Note that any linear subspace of $\mathbb{A}^{n+1}$ is an affine cone in particular $\mathbb{A}^{n+1}$ is an affine cone over $\mathbb{P}^n$.

Let $\hat{V} \subseteq \mathbb{A}^{n+1}$ be affine cone then projectivization of $\hat{V}$ is the set

$$\mathbb{P}(\hat{V}) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid (x_0, \ldots, x_n) \in \hat{V}\}$$

where $[x_0 : \cdots : x_n]$ are the homogeneous coordinates.

The point $x \in \mathbb{P}^n$ can be written in its homogeneous coordinates $x = [x_0 : \cdots : x_n]$. The affine representative of x is $\hat{x} \in \mathbb{A}^{n+1}$, so we have $x = [\hat{x}]$. Any two affine representatives of the same point differ by a nonzero scalar multiple. We call a subset $V \subseteq \mathbb{P}^n$ a projective subspace of dimension k, if there is a vector space $\hat{V} \subseteq \mathbb{A}^{n+1}$ of dimension $k+1$ such that $V = \mathbb{P}(\hat{V})$. In particular, we have $\dim(\mathbb{P}(\hat{V}))+1 = \dim V$. We use the convention that empty set $\emptyset$ is a projective subspace of dimension -1, that is $\dim(\emptyset) = -1$, and points $p \in \mathbb{P}^n$ are projective subspaces of dimension 0, that is $\dim(p) = 0$.

Given two projective subspaces V and W in $\mathbb{P}^n$, we write $V+W$ for the projective span (join) of V and W. It is defined to be the smallest projective subspace containing $V \cup W$. If however $\hat{V}$ and $\hat{W}$ are the linear subspaces of $\mathbb{A}^{n+1}$, we use the same notation for the direct sum $\hat{V} + \hat{W} \subseteq \mathbb{A}^{n+1}$.

Lemma 2.1. *Let V and W be projective subspaces of $\mathbb{P}^n$ such that $V \subseteq W$. Let A be another projective subspace of $\mathbb{P}^n$, then we have $V + A \subseteq W + A$.*

Proof. $V + A$ is defined to be smallest projective subspace containing $V \cup A$, we have $V \cup A \subseteq W \cup A \subseteq W + A$. Since $W + A$ is defined to be the smallest projective subspace containing $W \cup A$, and since any projective subspace containing $W \cup A$ must contain $V \cup A$, the minimal such subspace $V + A$ must be contained in $W + A$. ∎

Lemma 2.2. *For two linear subspaces $\hat{V}, \hat{W} \subseteq \mathbb{A}^{n+1}$, we have the following properties of projectivization:*

1. *(Monotonicity)* $\hat{V} \subseteq \hat{W}$ *implies* $\mathbb{P}(\hat{V}) \subseteq \mathbb{P}(\hat{W})$

2. *(Respects intersection)* $\mathbb{P}(\hat{V} \cap \hat{W}) = \mathbb{P}(\hat{V}) \cap \mathbb{P}(\hat{W})$

3. *(Respects sum)* $\mathbb{P}(\hat{V} + \hat{W}) = \mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W})$

Proof. We first show the monotonicity property. Let $x \in \mathbb{P}(\hat{V})$, then there is $\hat{x} \in \hat{V}$ such that $x = [\hat{x}]$. Since $\hat{V} \subseteq \hat{W}$, we have $\hat{x} \in \hat{W}$, hence $x \in \mathbb{P}(\hat{W})$.

To show the second claim, we first use the monotonic property of projectivization to show one direction. See that $\hat{V} \cap \hat{W} \subseteq \hat{V}$ and $\hat{V} \cap \hat{W} \subseteq \hat{W}$, therefore by taking projectivization of both of the subset relations and taking intersections, we get $\mathbb{P}(\hat{V} \cap \hat{W}) \subseteq \mathbb{P}(\hat{V}) \cap \mathbb{P}(\hat{W})$ which shows the one direction. For the other direction, let $x \in \mathbb{P}(\hat{V}) \cap \mathbb{P}(\hat{W})$, then there is nonzero $\hat{v} \in \hat{V}$ and $\hat{w} \in \hat{W}$ such that $x = [\hat{v}]$ and $x = [\hat{w}]$. Note that $\hat{v} = \lambda \hat{w}$ for some $\lambda \in \mathbf{k}^*$, therefore $\hat{v} \in \hat{W}$, hence $x = [\hat{v}] \in \mathbb{P}(\hat{V} \cap \hat{W})$.

Now for the last claim, we use the monotonic property of projectivization again. Since $\hat{V} \subseteq \hat{V} + \hat{W}$ and $\hat{W} \subseteq \hat{V} + \hat{W}$ by taking projectivization of both of the subset relations and joining them, we get $\mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W}) \subseteq \mathbb{P}(\hat{V} + \hat{W})$. For the other direction, let $x \in \mathbb{P}(\hat{V} + \hat{W})$ then $x = [\hat{v} + \hat{w}]$ for some $\hat{v} + \hat{w} \in \hat{V} + \hat{W}$. Then

$$[\hat{v}] \in \mathbb{P}(\hat{V}) \subseteq \mathbb{P}(\hat{V}) \cup \mathbb{P}(\hat{W}) \subseteq \mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W})$$

and

$$[\hat{w}] \in \mathbb{P}(\hat{W}) \subseteq \mathbb{P}(\hat{V}) \cup \mathbb{P}(\hat{W}) \subseteq \mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W}).$$

Furthermore since $\mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W})$ is defined to be smallest projective subspace containing $\mathbb{P}(\hat{V}) \cup \mathbb{P}(\hat{W})$ there is a linear subspace $\hat{U} \subseteq \mathbb{A}^{n+1}$ such that $\mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W}) = \mathbb{P}(\hat{U})$. Therefore, $\hat{v}, \hat{w} \in \hat{U}$ hence $x = [\hat{v} + \hat{w}] \in \mathbb{P}(\hat{U}) = \mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W})$. ■

A lemma that we are going to use throughout the thesis is called the Grassmann Formula.

Lemma 2.3 (Grassmann Formula). *For two projective subspaces V and W in $\mathbb{P}^n$, we have the identity:*

$$\dim(V) + \dim(W) = \dim(V + W) + \dim(V \cap W)$$

Proof. Since V and W are projective subspaces of $\mathbb{P}^n$, there has to be $\hat{V}$ and $\hat{W}$ linear subspaces of $\mathbb{A}^{n+1}$ such that $V = \mathbb{P}(\hat{V})$ and $W = \mathbb{P}(\hat{W})$. Furthermore using the properties of projectivization we have

$$V + W = \mathbb{P}(\hat{V}) + \mathbb{P}(\hat{W}) = \mathbb{P}(\hat{V} + \hat{W})$$

and

$$V \cap W = \mathbb{P}(\hat{V}) \cap \mathbb{P}(\hat{W}) = \mathbb{P}(\hat{V} \cap \hat{W}).$$

Now, if we apply the second isomorphism theorem of vector spaces for $\hat{V}$ and $\hat{W}$ we have the following isomorphism

$$\frac{\hat{V}}{\hat{V}\cap\hat{W}} \simeq \frac{\hat{W}+\hat{V}}{\hat{W}}.$$

Therefore, on dimensions we have $\dim(\hat{V})-\dim(\hat{V}\cap\hat{W}) = \dim(\hat{W}+\hat{V})-\dim(\hat{W})$. By rearranging, we get

$$\dim(\hat{V}) + \dim(\hat{W}) = \dim(\hat{W}+\hat{V}) + \dim(\hat{V}\cap\hat{W}).$$

Now see that

$$\begin{aligned}
\dim(V) + \dim(W) &= \dim \mathbb{P}(\hat{V}) + \dim \mathbb{P}(\hat{W}) \\
&= \dim(\hat{V}) - 1 + \dim(\hat{W}) - 1 \\
&= \dim(\hat{W}+\hat{V}) - 1 + \dim(\hat{V}\cap\hat{W}) - 1 \\
&= \dim \mathbb{P}(\hat{W}+\hat{V}) + \dim \mathbb{P}(\hat{V}\cap\hat{W}) \\
&= \dim(W+V) + \dim(V\cap W)
\end{aligned}$$

which is exactly what we wanted to show. ■

Similarly, a fact on on homogeneous polynomials that we are going to use several time is Euler's relation.

Proposition 2.4 (Euler's Relation). *Let $F \in \mathbf{k}[x_0, x_1, \dots, x_n]$ homogeneous of degree d. Then we have the following equality.*

$$\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = d \cdot F.$$

Proof. It is enough to prove the statement for a monomial $F = x_0^{\alpha_0} \dots x_n^{\alpha_n}$, where $\sum_{i=0}^{n} \alpha_i = d$ and α_i are nonnegative integers, because taking partial derivatives and multiplication distributes over sum, and any homogeneous polynomial of degree d can be written as a finite sum of monomials of degree d. The monomial case holds from a direct computation $\frac{\partial F}{\partial x_i} = \alpha_i x_0^{\alpha_0} \dots x_i^{\alpha_i - 1} \dots x_n^{\alpha_n}$ therefore $x_i \frac{\partial F}{\partial x_i} = \alpha_i F$ hence $\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = (\alpha_0 + \dots + \alpha_n) F = d \cdot F$. ■

For the rest of the thesis $X \subseteq \mathbb{P}^n$ will always be a smooth irreducible projective variety. Most of the statements are proved in more generality in their respective papers; however, we will keep X smooth and irreducible to achieve a better geometric understanding. The defining equations of $X = \mathbb{V}(F_1, \dots, F_s)$ where F_i are homogeneous polynomials for each i. The Jacobian matrix of X denoted as $\mathcal{J}_X$ is

$$\mathcal{J}_X = \begin{pmatrix} \frac{\partial F_1}{\partial x_0} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_s}{\partial x_0} & \cdots & \frac{\partial F_s}{\partial x_n} \end{pmatrix}.$$

Note that the Jacobian matrix $\mathcal{J}_X$ is a matrix of polynomials of size $s \times (n+1)$. For a point $p \in X$ and any affine representative $\hat{p}$ of p, we can evaluate the Jacobian matrix at $\hat{p}$ to get a matrix with entries in $\mathbf{k}$. Because the defining equations F_i are all homogeneous, replacing $\hat{p}$ with $\lambda\hat{p}$ does not affect the row span; hence, the kernel $\ker \mathcal{J}_X(\hat{p})$ only depends on the homogeneous coordinate $p = [\hat{p}]$. The tangent space of X at $p \in X$ denoted as T_pX is given by projectivization of the kernel of the Jacobian matrix of X evaluated at the point p. That is, $T_pX = \mathbb{P}(\ker \mathcal{J}_X(p))$, here we evaluate the Jacobian matrix $\mathcal{J}_X$ at the point $\hat{p}$ for any representative of p.

Lemma 2.5. *Let $p \in X$ then we have $p \in T_pX$.*

Proof. Since $p \in X$ we have $F_i(p) = 0$ for each $i \in [s]$. Let $i \in [s]$ be arbitrary, since F_i is homogeneous say degree d_i, then by Euler's relation 2.4 we have $\nabla F_i \cdot x = d_i F_i$ where $\cdot$ denotes the dot product. Now evaluating at $\hat{p}$ for any representative of p, we see that $\nabla F_i(\hat{p}) \cdot \hat{p} = 0$. Since i was arbitrary we see that $\hat{p} \in \ker \mathcal{J}_X(\hat{p})$ hence $p = [\hat{p}] \in T_pX$. ■

To visualize our projective varieties X, we are going to look at their affine chart. Let H_∞ be the hyperplane at infinity, although it can change in all our examples it will be given by $H_\infty = \mathbb{V}(x_0)$. We denote $\mathbb{A}^n := \mathbb{P}^n \setminus H_\infty$ to be the affine chart where $x_0 \neq 0$ and we identify the points with $x_0 = 1$ by dehomogenization. We will denote $Y := X \cap \mathbb{A}^n$ to be the affine part of the projective variety. The defining equations of Y are $\mathbb{V}(f_1, \ldots, f_s)$ where f_i is dehomogenized polynomial F_i, that is $f_i(z_1, \ldots, z_n) = F_i(1, z_1, \ldots, z_n)$. Note that if $X \not\subseteq H_\infty$ then $Y \neq \emptyset$, therefore we will assume that X is not contained in the hyperplane at infinity. Furthermore, if X is smooth, then Y is also smooth.

We recall some basic linear algebra facts that will be used throughout the thesis. We start with the rank–nullity theorem; a proof can be found in any standard linear algebra textbook.

Theorem 2.6 (Rank-Nullity Theorem)**.** *Let $\hat{U}$ and $\hat{V}$ be $\mathbf{k}$-vector spaces and let $f : \hat{U} \to \hat{V}$ be a $\mathbf{k}$-linear transformation then*

$$\dim(\hat{U}) = \dim(\ker(f)) + \dim(\operatorname{Im}(f))$$

The following lemma is going to enable us to show equivalent definitions of polar varieties.

Lemma 2.7. *Let $\hat{U}, \hat{V}, \hat{W}$ be $\mathbf{k}$-vector spaces and $f : \hat{U} \to \hat{V}$ and $g : \hat{V} \to \hat{W}$ be two linear transformations then we have both*

$$\dim(\operatorname{Im} f) = \dim(\operatorname{Im}(g \circ f)) + \dim(\ker g \cap \operatorname{Im} f) \tag{1}$$

and

$$\dim(\ker(g \circ f)) = \dim(\ker f) + \dim(\ker g \cap \operatorname{Im} f) \tag{2}$$

Proof. To show the first relation (1), we first define $g' : \operatorname{Im} f \to \hat{W}$ as the restriction of g on the image of f, then see that $\ker g' = \ker g \cap \operatorname{Im} f$ and $\operatorname{Im} g' = \operatorname{Im}(g \circ f)$. Now using rank nullity theorem 2.6 on g' we get

$$\dim(\operatorname{Im} f) = \dim(\ker g') + \dim(\operatorname{Im} g') = \dim(\ker g \cap \operatorname{Im} f) + \dim(\operatorname{Im}(g \circ f)).$$

To show the second relation (2), we use rank nullity theorem 2.6 on both f and $g \circ f$ to get the equations

$$\dim U = \dim(\ker f) + \dim(\operatorname{Im} f)$$

and

$$\dim U = \dim(\ker(g \circ f)) + \dim(\operatorname{Im}(g \circ f)).$$

now using (1) and by equating the relations on $\dim U$ we get

$$\begin{aligned} \dim(\ker(g \circ f)) &= \dim(\ker f) + \dim(\operatorname{Im} f) - \dim(\operatorname{Im}(g \circ f)) \\ &= \dim(\ker f) + \dim(\ker g \cap \operatorname{Im} f) \end{aligned}$$

which concludes the proof. ■

Finally, since we are going to write projective subspaces as the projectivization of the image of a linear transformation, we frequently use the characterization of the rank conditions of matrices via minors.

Theorem 2.8 (Matrix Rank Condition via Minors)**.** *Let A be a $n \times m$ matrix with entries in a field* $\mathbf{k}$*. Let $c \leq \min(n, m)$, then we have $rk(A) < c$, if and only if all $c \times c$ minors vanish.*

Proof. ($\Longrightarrow$) Take arbitrary c columns from the matrix A. Since $\operatorname{rk} A < c$, the c columns must be linearly dependent. Any c rows of these c columns, continue to be linearly dependent. So corresponding $c \times c$ submatrix has determinant 0. Since, we have chosen arbitrary c columns and c rows, it must be the case that all $c \times c$ minors vanish.

($\Longleftarrow$) We prove the contrapositive of the statement. Assume that $\operatorname{rk} A \geq c$, then there should be at least c columns of A such that they are linearly independent. Call the $n \times c$ submatrix A'. Since $c \leq n$, the submatrix A' is full rank, that is $\operatorname{rk} A' = c$, implies $\operatorname{rk}(A')^T = c$, therefore there must be c linearly independent columns of $(A')^T$. Therefore $c \times c$ submatrix of $(A')^T$ must have nonzero determinant. Since this $c \times c$ submatrix of $(A')^T$ is also a submatrix of A, we have found a nonzero $c \times c$ minor of A. ■

2.2 Classical Polar Varieties

For rest of the chapter, assume X is smooth irreducible projective variety in $\mathbb{P}^n$ where the field is $\mathbf{k} = \mathbb{C}$. We write the dimension $\dim X = m$ and codimension $\operatorname{codim} X = c$, and its defined by

$X = \mathbb{V}(F_1, \ldots, F_s)$. The first and the main definition of a polar variety with respect to a subspace is given as a set of points at the variety whose tangent space has a non-transversal intersections with the subspace. We recall the definition of a non-transversal intersection.

Definition 2.1. Let $V \subseteq \mathbb{P}^n$ be a projective subspace. We say that V intersects X *non-transversely* at $p \in X$, if $p \in V$ and $\dim(V + T_pX) < n$ and it is denoted as $V \not\pitchfork_p X$.

Example 2.1. Let X be a smooth curve in $\mathbb{P}^3$. Since it is one dimensional, the tangent space at any point of X is one dimensional. Then every line that intersects X does so non-transversely since the total dimension does not equal to the dimension of the ambient space. △

Definition 2.2. [10, Definition 4.2] The polar variety of $X \subseteq \mathbb{P}^n$ with respect to a subspace $V \subseteq \mathbb{P}^n$ is

$$P(X, V) := \{p \in X \mid V + p \text{ intersects } X \text{ at } p \text{ non-transversely } \}.$$

For generic projective subspace V of dimension $\dim V = c+i-2$ for some $i \in \{0, \ldots, m = \dim X\}$, the set $P(X, V)$ is called the i-th polar variety, it is denoted as $P_i(X, V)$ and its degree $\mu_i(X)$ is called the i-th polar degree. The reason we omit the projective subspace V in the notation of polar degree $\mu_i(X)$ is for a generic V the degree does not depend on V.

Remark 2.1. An equivalent way of writing $(V+p) \not\pitchfork T_pX$ which will be used throughout the thesis is

$$\dim(V \cap T_pX) > \dim V - \operatorname{codim} X. \tag{3}$$

This can be shown as follows

$$\begin{aligned} n &> \dim((V+p) + T_pX) \\ &= \dim(V + T_pX) \\ &= \dim V + \dim T_pX - \dim(V \cap T_pX), \end{aligned}$$

where we used the definition of non-transversal intersection, the Lemma 2.5 on $p \in T_pX$, Grassmann formula 2.3 and finally reordering.

Furthermore, for the i-th polar variety since $\dim V = c + i - 2$, the equation (3) becomes

$$\dim(V \cap T_pX) \geq i - 1. \tag{4}$$

We note that some authors define the polar varieties using equation (4) instead of non-transversal intersections; however, both definitions yield the same polar variety.

Example 2.2. [10, Example 4.4] We note that the polar variety for $i = 0$ is the variety X itself that is $P_0(X, V) = X$ in particular $\mu_0(X) = \deg X$. To see that, by equation (4), the polar variety equals to $P_0(X, V) = \{p \in X \mid \dim(V \cap T_pX) \geq -1.\}$. Since $\dim(V \cap T_pX) \geq -1$ holds for all points in $p \in X$, we have $P(X, V) = X$, hence $\mu_0(X) = \deg X$. △

Example 2.3. Let the ambient space be $\mathbb{P}^2$ with homogeneous coordinates $[x_0 : x_1 : x_2]$, let the projective variety be $X = \mathbb{V}(x_1^2 + x_2^2 - x_0^2) \subseteq \mathbb{P}^2$. In the affine chart $Y = X \cap \mathbb{A}^n = \mathbb{V}(x_1^2 + x_2^2 - 1)$ defines a circle with radius 1 centered around the origin. For $i = 0$, we have seen in the Example 2.2 that $P(X, V) = X$. For $i = 1$, the dimension of V is $\dim V = 0$, therefore the projective subspace V is a single point, which we denote as $v = [v_0 : v_1 : v_2] \in \mathbb{P}^2$. Note that since X is a hypersurface (a codimension 1 variety), the projective line spanned by v and p denoted as $v + p$ intersects X at p non-transversely if and only if $v + p$ is contained in T_pX.

We are looking for points $p \in X$ whose tangent space contains the line spanned by v and p, $v + p$. As shown in the lemma 2.5, p is already a point in T_pX, making this condition equivalent to v contained in T_pX. The equation for tangent space of X at p is the kernel of the row vector $\begin{pmatrix} -2p_0 & 2p_1 & 2p_2 \end{pmatrix}$. Therefore, the condition that $v \in T_pX$ can be given by the following system of equations:

$$-2p_0v_0 + 2p_1v_1 + 2p_2v_2 = 0 \tag{5}$$

$$p_1^2 + p_2^2 - p_0^2 = 0, \tag{6}$$

where (5) is the condition for v lies in the tangent space T_pX and the (6) ensures that $p \in X$.

Let us visualize the first polar variety in the affine chart, analyzing case by case when v_0 is nonzero and when it is zero.

For $v_0 \neq 0$, we can scale the coordinates without loss of generality to write $v = (1 : v_1 : v_2)$. For example for $v_1 = 2$ and $v_2 = 0$, the point is $v = (1 : 2 : 0)$, and the (5) becomes $-2p_0 + 4p_1 = 0$. In the affine chart $x_0 = 1$, this is the line $x_1 = \frac{1}{2}$ intersected with the circle $x_1^2 + x_2^2 = 1$ as shown in the Figure 1 below.

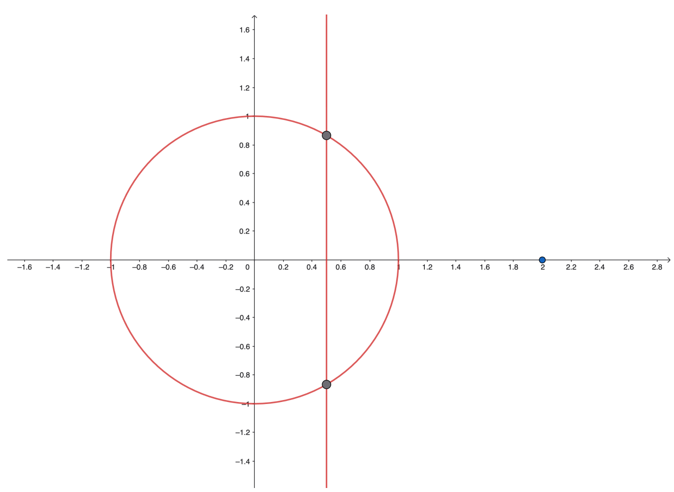

Figure 1: The circle intersecting with the line $x_1 = \frac{1}{2}$.

The points of intersection are exactly the two points in X which is shown in color grey whose tangent line passes through the purple point v. This is illustrated in the Figure 2 below:

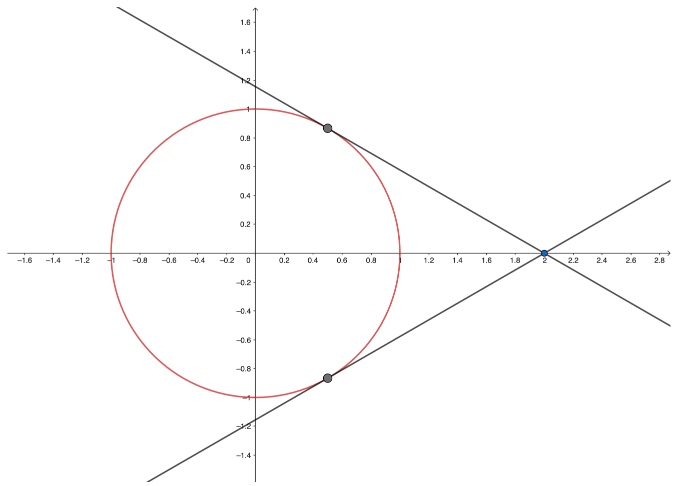

Figure 2: The first polar variety of X are the 2 points whose tangent spaces passes through v

If $v_0 = 0$, then $v = [0 : v_1 : v_2]$ is a point in hyperplane at infinity $H_\infty = \mathbb{V}(x_0)$, which corresponds to a direction in $\mathbb{A}^2$. For example if $v_1 = 1$ and $v_0 = 0$, the point $v = [0 : 1 : 0]$ represents the x_1 direction in $\mathbb{A}^2$. The tangency equation (5) then reduces to $p_1 = 0$, which means we are intersecting X with $x_1 = 0$ giving two points in the circle colored in red, as seen in the Figure 3 below:

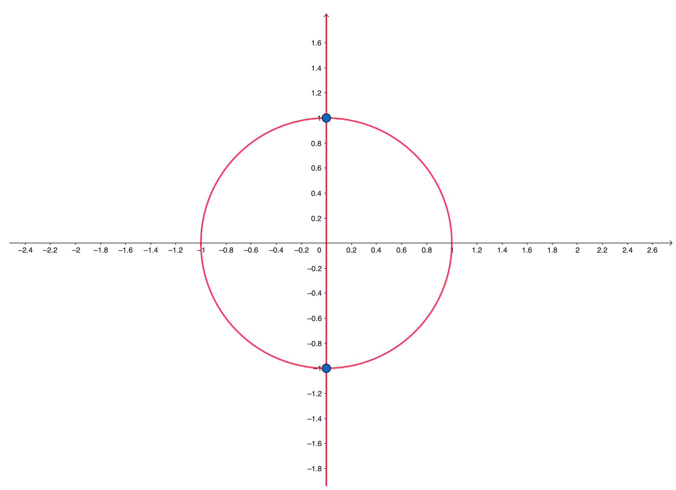

Figure 3: The circle intersecting with the line $x_1 = 0$

This can also be thought as the point (v_1, v_2) reaches to infinity in the x direction. Therefore the polar variety is capturing the directions of the the tangent space of the variety.

To better understand the underlying geometry, we can visualize $\mathbb{P}^2$ via its affine cone $\mathbb{A}^3$. In all of the following figures, the axes z, x, y corresponds to x_0, x_1, x_2 respectively.

In the Figure 4 the point $v = [1 : 2 : 0]$ corresponds to the line passing through the origin in $\mathbb{A}^3$. Since projective span of two points translates to an affine plane in the affine cone, the first polar variety is the points on X whose projective tangent spaces passing through the v.

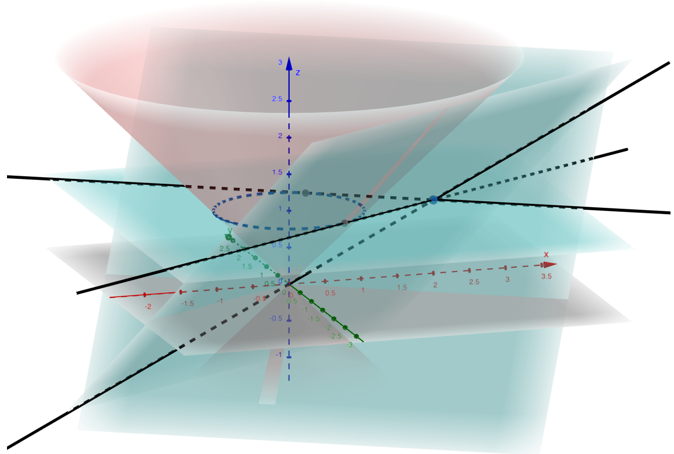

Figure 4: Affine cone of $\mathbb{P}^2$ with the tangent spaces of X and the point v is visible.

In the Figure 4 the cone centered at the origin is the affine cone for the $X = \mathbb{V}(x_1^2 + x_2^2 - x_0^2)$. The black line through the origin passing through the blue point is the line $(1:2:0)$. The plane where the blue point lives in in is the $z = 1$, that is $x_0 = 1$ plane. The section of this plane is exactly the Figure 2.

Similarly, Figure 5 illustrates the case where the point v lies in the hyperplane at infinity. Here, the polar variety captures the points in X whose tangent spaces are parallel to the direction of v.

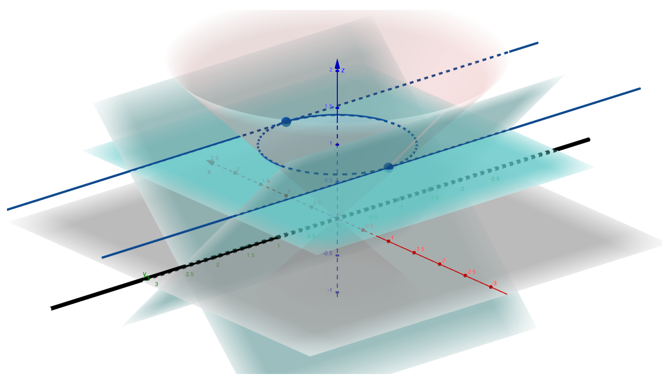

Figure 5: Affine cone of $\mathbb{P}^2$ with the tangent spaces of X and the point v is visible.

Here in this picture the black line passing through the origin is the direction $v = (0:1:0)$ and the two blue points on the circle is the first polar variety. Note that the tangent space of X at the blue points as shown as planes in the figure is containing the black line v.

In summary, this examples demonstrates two different geometric behaviors of polar variety. When the projective subspace V is contained in the hyperplane at infinity, the polar variety identifies points on X whose tangent spaces have a specific direction. When V lies in the affine plane, the polar variety consists of points whose tangent spaces pass directly through v. $\triangle$

When we pass from the $\mathbb{P}^2$ to the $\mathbb{P}^3$ the affine cone picture is harder to visualize since it is $\mathbb{A}^4$, however the picture for polar variety in the affine chart becomes more clear.

Example 2.4 (Ellipsoid in $\mathbb{P}^3$)**.** Let $X = \mathbb{V}(c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - x_0^2) \subseteq \mathbb{P}^3$, and V be a projective subspace in $\mathbb{P}^3$. We consider the i-th polar varieties $P(X, V)$ where $i = 0, 1, 2$.

For the case $i = 0$, we have already seen from Example 2.2 that 0-th polar variety is always the X itself.

For the cases where $i \neq 0$, similar to the discussion in Example 2.3, since X is a hypersurface, $V + p$ intersects X at p non-transversely if and only if $V + p \subseteq T_pX$. However, since $p \in T_pX$ this condition simplifies to $V \subseteq T_pX$. The projective tangent space T_pX at a point $p \in X$ is given by the kernel of the gradient row vector $\begin{pmatrix} -2p_0 & 2c_1p_1 & 2c_2p_2 & 2c_3p_3 \end{pmatrix}$. Therefore, the defining equation for T_pX is

$$-p_0x_0 + c_1p_1x_1 + c_2p_2x_2 + c_3p_3x_3 = 0.$$

For $i = 1$ the dimension of V is $\dim V = 0$. Therefore, the projective subspace V is a single point, denoted as $v = [v_0 : v_1 : v_2 : v_3] \in \mathbb{P}^3$. The condition $V \subseteq T_pX$ simply means $v \in T_pX$ in this case. Therefore, the first polar variety $P(X, v)$ is given by the system of equation

$$\begin{aligned} -v_0p_0 + c_1v_1p_1 + c_2v_2p_2 + c_3v_3p_3 &= 0 \\ c_1p_1^2 + c_2p_2^2 + c_3p_3^2 - p_0^2 &= 0. \end{aligned}$$

This is an intersection of hyperplane with an ellipsoid therefore for generic v, this system defines a curve contained in ellipsoid X. We can analyze the curve by considering the case that v lies in the H_∞ or in the affine chart $\mathbb{A}^3$.

In all of the following figures, we took $c_1 = 1, c_2 = 2, c_3 = 3$ and the axes x, y, z corresponds to x_1, x_2, x_3 respectively.

If v lies in the affine chart that is $v_0 \neq 0$, for example $v = (1 : 2 : 0 : 0)$ then the first polar variety of X is the curve defined by intersecting $x_1 = \frac{1}{2c_1}$ with the ellipsoid.

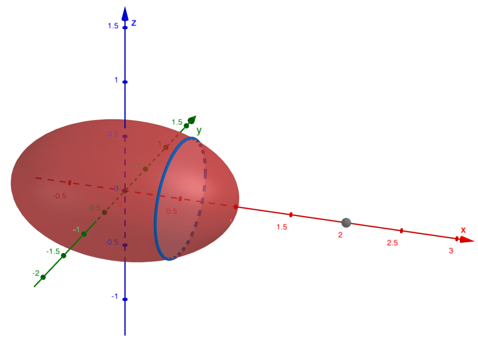

Figure 6: The first polar variety of X with respect to an affine point v.

The gray point on the x-axis is the $v = (1 : 2 : 0 : 0)$ in the affine chart. The blue curve contained in the ellipsoid is the first polar variety $P(X, v)$. Note that for each point in the blue curve which is the first polar variety the tangent planes are passing through the point v as shown in the figure below.

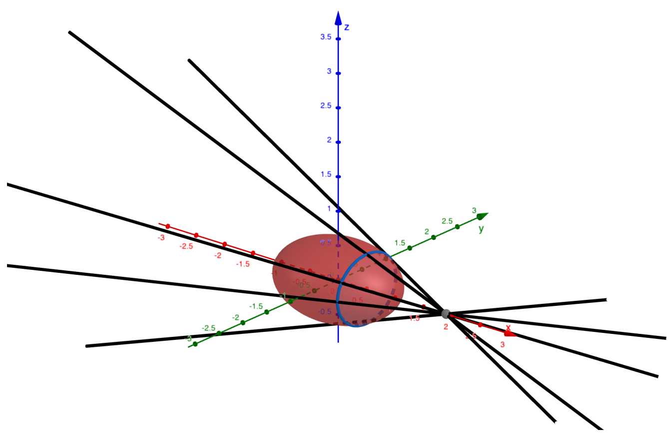

Figure 7: The first polar variety of X with respect to an affine point v. The curve consists of the points on X whose tangent planes passing through v

If v lies in the hyperplane at infinity $v_0 = 0$, then it represents a direction in $\mathbb{A}^3$. The first polar variety is a curve consists of all the points on the ellipsoid whose tangent plane is parallel to the direction of v, as shown below.

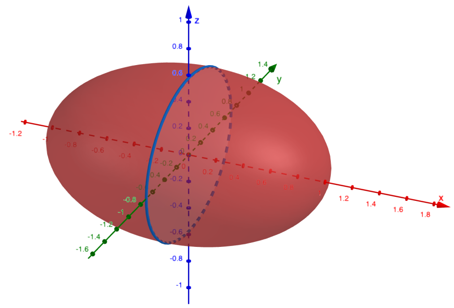

Figure 8: The first polar variety of X with respect to an affine point $v \in H_\infty$.

For the case $i = 2$, the dimension of V is $\dim V = 1$; therefore, the projective subspace V is a line $L \subseteq \mathbb{P}^3$. A line L in $\mathbb{P}^3$ is determined by the projective span of two distinct points $v = [v_0 : v_1 : v_2 : v_3]$ and $w = [w_0 : w_1 : w_2 : w_3]$. To satisfy the condition $L \subseteq T_pX$, both the basis points v and w must lie in T_pX. Hence, second polar variety $P(X, L)$ is given by the system equations.

$$\begin{aligned} -v_0p_0 + c_1v_1p_1 + c_2v_2p_2 + c_3v_3p_3 &= 0 \\ -w_0p_0 + c_1w_1p_1 + c_2w_2p_2 + c_3w_3p_3 &= 0 \\ c_1p_1^2 + c_2p_2^2 + c_3p_3^2 - p_0^2 &= 0. \end{aligned}$$

First two equation define a line in $\mathbb{P}^3$, whose intersection with ellipsoid consists of two points.

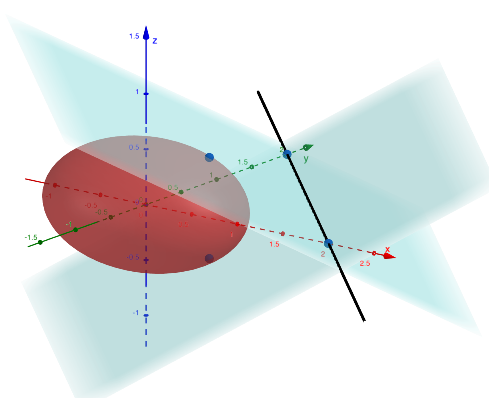

Figure 9: The second polar variety of X with respect to an affine line L

Here the black line passing through v and w, is the L in the affine chart. The blue points on the ellipsoid are the two discrete points representing the second polar variety. Note that tangent space of the ellipsoid in blue points are exactly the points whose tangent plane is containing L.

Furthermore if we combine the figures of all the polar varieties we can see their subset inclusions as well.

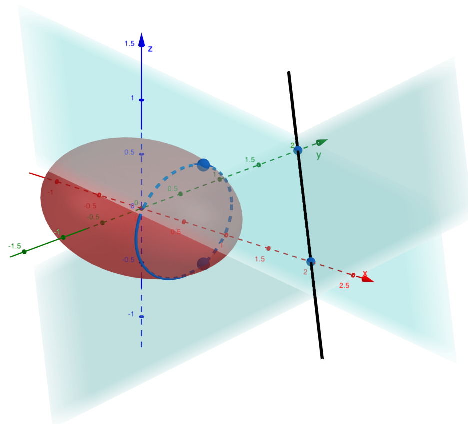

Figure 10: $P(X,L) \subseteq P(X,v)$ when $v \in L$

The geometric picture we got is depending on where the projective subspace V lives, just like in the Example 2.3 polar variety $P(X,V)$ acts the same way. If V lies in the affine chart, we are looking for tangent spaces containing V. If V lies in the hyperplane at infinity, $P(X,V)$ identifies points on X whose tangent space have the same direction given by V. $\triangle$

Example 2.5 (Polar degrees of smooth hypersurfaces)**.** Let $X = \mathbb{V}(F) \subseteq \mathbb{P}^n$ be the smooth projective variety where F is a homogeneous polynomial of degree d. Since X is a hypersurface, it is codimension $c = 1$, so for a generic $V \subseteq \mathbb{P}^n$ projective subspace of dimension $\dim V = c+i-2 = i-1$ we are going to calculate the polar degrees $\mu_i(X)$ for $i \in \{0,\dots,\dim X\}$, therefore fix an $i \in \{0,\dots,\dim X\}$.

For $p \in X$ since $\dim T_pX = n-1$, the non-transversal intersection $V + p \not\pitchfork_p T_pX$ happens if and only if $V + p \subseteq T_pX$. Since $p \in T_pX$ by Lemma 2.5, we are looking for the conditions $V \subseteq T_pX$. Since the dimension of $\dim V = i-1$, it is given by projectivization of span of i linearly independent vectors, say $\{v_1,\dots,v_i\}$. The condition $v_j \in T_pX$ if and only if $\nabla_{v_j}F(p) = 0$ for $j \in [i]$, where $\nabla_{v_j}F$ is the directional derivative of F in the direction v_j. Hence the i-th polar variety $P_i(X,V)$ is

$$P_i(X,V) = \{p \in X \mid \forall j \in [i],\ \nabla_{v_j}F(p) = 0\},$$

furthermore it is given by the homogeneous ideal

$$\langle F, \nabla_{v_1}F, \dots, \nabla_{v_i}F \rangle .$$

Note that $\deg F = d$ and $\deg \nabla_{v_j}F = d-1$. Since V is generic, the points $\{v_1,\dots,v_i\}$ are also generic, therefore the hypersurfaces $\nabla_{v_j}F$ intersect X properly. Therefore by Bezout's theorem the degree of $P_i(X,V)$ is $d(d-1)^i$. Therefore $\mu_i(X) = d(d-1)^i$. △

For an example where X is not a hypersurface, we are going to wait until the Chapter 2.5 to have a computational approach to define polar varieties. Now that we have geometric intuition of what classical polar varieties are capturing, we are going to see other characterizations of it in the following chapters.

2.3 Classical Polar Varieties as a pullback through Gauss map

We give another definition of polar varieties as a pullback of a special Schubert variety via Gauss map. We first recall the Grassmannian to define the Schubert varieties and define the Gauss map.

Definition 2.3. The Grassmannian $Gr(k,\mathbb{P}^n)$ is the set of all k dimensional projective subspaces of $\mathbb{P}^n$.

Remark 2.2. Since we have already discussed that the projective subspaces are given by projectivization of linear subspaces in the affine cone, we can also work with $Gr(k+1,\mathbb{A}^{n+1})$, which parametrizes the $k+1$ dimensional linear subspaces of $\mathbb{A}^{n+1}$, differing only by identification with projectivization.

Example 2.6 (Projective Space)**.** When $k=0$, the Grassmannian coincides with the projective space $Gr(1,\mathbb{A}^{n+1}) = \mathbb{P}^n$. △

Example 2.7 (Dual Projective Space)**.** When $k=n-1$ the points in the Grassmannian $Gr(n-1,\mathbb{P}^n) = Gr(n,\mathbb{A}^{n+1})$ are the hyperplanes in $\mathbb{P}^n$. In the literature, this is called the *dual projective space* $(\mathbb{P}^n)^\vee$. △

Proposition 2.9. *[1, Proposition 14.2] Let Θ' and Θ be two full rank matrices of dimension $(k+1)\times(n+1)$ where $k \le n$. Then the equality* $\mathrm{row}(\Theta') = \mathrm{row}(\Theta)$ *holds if and only if there exists a scalar $\gamma \in \mathbf{k}$ such that, for every $\sigma \subseteq [n+1]$ with $|\sigma| = k+1$, we have* $\det(\Theta_\sigma) = \gamma\det(\Theta'_\sigma)$*. Where Θ_σ denotes the $(k+1)\times(k+1)$ submatrix of Θ whose columns are given by σ.*

Definition 2.4. Since every projective subspace $V \subseteq \mathbb{P}^n$ can be written as a projectivization of the row span of a full rank matrix Θ of size $(k+1)\times(n+1)$, and by the content of the proposition, the row span is characterized by the maximal minors up to some scalars. There we have $G(k,\mathbb{P}^n) \subseteq \mathbb{P}^{\binom{n+1}{k+1}-1}$.

Given $(k+1)$-dimensional subspace $\hat{V}_{k+1} \in G(k+1,\mathbb{A}^{n+1})$, the maximal minors $\det(\Theta_\sigma)$ of a $(k+1)\times(n+1)$ matrix Θ with $\mathrm{row}(\Theta) = \hat{V}_{k+1}$ are called the *Plücker coordinates* for $\hat{V}_{k+1}$. There are $\binom{n+1}{k+1}$ many maximal minors of a $(k+1)\times(n+1)$ matrix, and they represent the subspace $\hat{V}_{k+1}$ up to a scalar factor $\gamma \in \mathbf{k}$ as in Proposition 2.9.

Definition 2.5. Denote the $(k+1)\times(n+1)$ matrix of variables by $\mathbf{x} = (x_{ij})$, where $i \in [k+1]$ and $j \in [n+1]$. The subalgebra of $\mathbf{k}[\mathbf{x}]$ generated by the $\det(\mathbf{x}_\sigma)$ for $\sigma \in \binom{[n+1]}{k+1}$ is called the Plücker algebra. Also, denote by $\mathbf{k}[\mathbf{p}]$, the algebra where $\mathbf{p}_\sigma$ is a variable for each $\sigma \in \binom{[n+1]}{k+1}$. Define the ring homomorphism

$$\phi_{n+1} : \mathbf{k}[\mathbf{p}] \to \mathbf{k}[\mathbf{x}], \quad p_\sigma \mapsto \det(\mathbf{x}_\sigma)$$

then the Plücker algebra is the $\mathbf{k}[\mathbf{p}]/\ker\phi_{n+1}$. We can further identify each $\sigma \subseteq [n+1]$ with $|\sigma| = k+1$ with an ordered string in $\mathbf{k}[\mathbf{p}]$.

Example 2.8. For the $Gr(2, \mathbb{A}^4)$ we have 8 variables and we can think of them in terms of 2×4 matrix

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

For each $\sigma \subseteq [4]$, with $|\sigma| = 2$ we consider the maximal minor as p_σ. For instance $p_{12} = x_{11}x_{22} - x_{12}x_{21}$ and $p_{21} = -p_{12}$. All the Plücker coordinates will be

$$(x_{11}x_{22} - x_{21}x_{12} : \cdots : x_{13}x_{24} - x_{23}x_{14}) \in \mathbb{P}^{\binom{4}{2}-1}$$

The graph of ϕ_n lies in $\mathbf{k}^6 \times \mathbf{k}^8$ and it is given by the parametrization

$$\begin{aligned} p_{12} - (x_{11}x_{22} - x_{21}x_{12}) &= 0 \\ &\vdots \\ p_{34} - (x_{13}x_{24} - x_{23}x_{14}) &= 0. \end{aligned}$$

If we take the ideal of it and project it down to $\mathbf{k}^6$, this is the same as elimination, and essentially it records all the relations involving only the variables p_{12} through p_{34}. Hence we get the ideal $\langle p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14}\rangle$ which is called the Plücker relations for $Gr(2, \mathbb{A}^4)$. $\triangle$

Example 2.9. Recall that in projective space, the representation of homogeneous coordinates is not unique, unless we pass to one of the affine charts. For the point $(x_0 : \cdots : x_n) \in \mathbb{P}^n$, we normalize a term $x_i \neq 0$ by setting $x_i = 1$; then the representation becomes unique.

First of all since $\hat{V}_2 \in Gr(2, \mathbb{A}^4)$ is 2-dimensional subspace, the matrix Θ has rank 2. Therefore, there must be a 2×2 minor that is nonzero. For simplicity, assume $p_{12} \neq 0$.

The counterpart of normalizing a term in projective space for the Plücker coordinates in the Grassmannian is choosing the representation of the matrix with $p_{12} = 1$. This means that the first maximal minor is 1; hence, the matrix can be written in the form

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Note that this is the reduced row echelon form of the 2×4 matrix which is a unique representation. Now from this, we get the equation

$$p_{34} = ad - bc = (-p_{23})p_{14} - (-p_{24})p_{13}.$$

If we homogenize this relation with p_{12}, we get the equation $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$ again. $\triangle$

Now that we have defined the Grassmannian, we define subvarieties of it which are called Schubert varieties. We follow [19, Chapter 4.1].

Definition 2.6 (Complete Flag in $\mathbb{A}^n$)**.** A complete flag $\hat{\mathcal{V}}$ in $\mathbb{A}^n$ is a sequence of projective subspaces

$$\hat{\mathcal{V}} : 0 = \hat{V}_0 \subset \hat{V}_1 \subset \cdots \subset \hat{V}_{n-1} \subset \mathbb{A}^n.$$

with $\dim \hat{\mathcal{V}}_i = i$.

Definition 2.7 (Schubert varieties in $\mathbb{A}^n$)**.** Let $\hat{\mathcal{V}}$ be complete flag in $\mathbb{A}^n$, let $\mathbf{a} = (a_1, \ldots, a_k)$ be a sequence of integers such that $n - k \geq a_1 \geq \cdots \geq a_k \geq 0$, then the set

$$\Sigma_{\mathbf{a}} = \left\{ \Lambda \in Gr(k, \mathbb{A}^n) \mid \dim \left(\hat{V}_{n-k+i-a_i} \cap \Lambda \right) \geq i \text{ for all } i = 1, \ldots, k \right\}$$

is called Schubert variety.

Since X is a projective variety we are going to change the indices accordingly since the correspondence was $Gr(k, \mathbb{P}^n) = Gr(k+1, \mathbb{A}^{n+1})$.

Definition 2.8 (Complete Flag in $\mathbb{P}^n$)**.** A complete flag $\mathcal{V}$ in $\mathbb{P}^n$ is a sequence of projective subspaces

$$\mathcal{V} : \emptyset = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{P}^n$$

where $\dim V_i = i$.

Definition 2.9 (Projective Schubert variety)**.** Let $\mathcal{V}$ be a complete flag in $\mathbb{P}^n$. Let $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ be a non-increasing sequence of integers such that $n - k \geq a_0 \geq a_1 \geq \cdots \geq a_k \geq 0$. The Schubert cycle $\Sigma_{\mathbf{a}}$ in Grassmannian $Gr(k, \mathbb{P}^n)$ is defined as

$$\Sigma_{\mathbf{a}} = \{\Lambda \in Gr(k, \mathbb{P}^n) \mid \dim(V_{n-k+j-a_j} \cap \Lambda) \geq j \text{ for all } j = 0, 1, \ldots, k\}.$$

Now going back to discussions on the polar varieties, recall that by the remark 2.1 we have $V + p$ intersects X at $p \in X$ non-transversely if and only if

$$\dim \left(V \cap T_p X \right) > \dim V - \operatorname{codim} X.$$

Furthermore, the expected codimension of the intersection of two projective subspaces V and T_pX is $\operatorname{codim} V + \operatorname{codim} T_pX$, therefore expected dimension of the intersection is equal to

$$n - (\operatorname{codim} V + \operatorname{codim} T_pX) = n - (n - \dim V) - \operatorname{codim} T_pX = \dim V - \operatorname{codim} X.$$

Hence, we can rephrase the condition of $V + p$ intersects X at $p \in X$ non-transversely to be intersection of V and T_pX with higher than the expected dimension.

We can record all projective subspaces that intersects V in an unexpected manner as a subvariety of the Grassmannian

$$\Sigma_m(V) = \{\Lambda \in Gr(m, \mathbb{P}^n) \mid \dim(V \cap \Lambda) > \dim V - \operatorname{codim} \Lambda\}.$$

Note that assuming the dimension of V to be $\dim V = c + i - 2$ and since $\operatorname{codim} \Lambda = n - m = c$ we can rewrite the set as

$$\Sigma_m(V) = \{\Lambda \in Gr(m, \mathbb{P}^n) \mid \dim(V \cap \Lambda) \geq i - 1\}.$$

Therefore the condition of $V + p$ intersects X at $p \in X$ non-transversely, is the statement where $T_pX \in \Sigma_m(V)$. These subvarieties $\Sigma_m(V)$ of Grassmannian $G(m, \mathbb{P}^n)$ are called *special Schubert varieties* and we show exactly which one they are in the lemma below.

Lemma 2.10. *Let* $X \subseteq \mathbb{P}^n$*, denote* $\dim X = m$, $\operatorname{codim} X = c$*, fix* $i \in \{1, \ldots, m\}$*, and let* $\dim V = c + i - 2$*. Let* $\mathcal{V}$ *be a complete flag in* $\mathbb{P}^n$ *such that it contains* V *as well. The Schubert variety*

$$\Sigma_{\mathbf{a}} = \{\Lambda \in Gr(m, \mathbb{P}^n) \mid \dim(V_{c+j-a_j} \cap \Lambda) \geq j \textit{ for } j = 0, \ldots, m\}$$

for sequence $\mathbf{a} = (\underbrace{1, \ldots, 1}_{i \text{ many}}, 0, \ldots, 0)$ *is the same as the set*

$$\Sigma_m(V) = \{\Lambda \in Gr(m, \mathbb{P}^n) \mid \dim(V \cap \Lambda) \geq i - 1\}.$$

Proof. First of all note that since $\dim V = c + i - 2$, we write $V = V_{c+i-2}$ for the element in the flag $\mathcal{V}$.

The inclusion $\Sigma_{\mathbf{a}} \subseteq \Sigma_m(V)$ is straightforward since for $j = i - 1$, we get $\dim(V \cap \Lambda) \geq i - 1$. We show the other inclusion $\Sigma_m(V) \subseteq \Sigma_{\mathbf{a}}$ case by case on j. Fix $\Lambda \in \Sigma_m(V)$ and $j \in \{1, \ldots, m\}$.

If $j = i - 1$, they correspond to the same condition, therefore there is nothing to show.

If $j > i-1$ then $a_j = 0$, we use the Grassmann formula 2.3:

$$\begin{aligned}\dim(V_{c+j} \cap \Lambda) &= \dim(V_{c+j}) + \dim(\Lambda) - \dim(V_{c+j} + \Lambda)\\ &\geq (c+j) + m - n\\ &= j,\end{aligned}$$

where we used the fact that $V_{c+j} + \Lambda \subseteq \mathbb{P}^n$ in the inequality.

If $j < i-1$, then $a_j = 1$, and we have $V_{c+j-1} \subseteq V_{c+i-2}$. We need to show that $\dim(V_{c+j-1} \cap \Lambda) \geq j$. The claim is $\dim(V_{c+j-1} + \Lambda) \leq n-1$. After proving that the desired result can be shown by using Grassmann formula 2.3 to

$$\begin{aligned}\dim(V_{c+j-1} \cap \Lambda) &= \dim(V_{c+j-1}) + \dim(\Lambda) - \dim(V_{c+j-1} + \Lambda)\\ &\geq (c+j-1) + m - (n-1)\\ &= j.\end{aligned}$$

To prove the claim $\dim(V_{c+j-1} + \Lambda) \leq n-1$, we use the assumption of $\dim(V_{c+i-2} \cap \Lambda) \geq i-1$ coming from $\Lambda \in \Sigma_m(V)$ and the fact that $V_{c+j-1} \subseteq V_{c+i-2}$ from the condition $j < i-1$. Now using Grassmann formula 2.3 we get

$$\begin{aligned}\dim(V_{c+i-2} + \Lambda) &= \dim V_{c+i-2} + \dim \Lambda - \dim(V_{c+i-2} \cap \Lambda)\\ &\leq (c+i-2) + m - (i-1)\\ &= n-1.\end{aligned}$$

Now since $V_{c+j-1} \subseteq V_{c+i-2}$, we get $V_{c+j-1} + \Lambda \subseteq V_{c+i-2} + \Lambda$ as well, therefore after taking dimensions we get

$$\dim(V_{c+j-1} + \Lambda) \leq \dim(V_{c+i-2} + \Lambda) \leq n-1$$

proving our claim $\dim(V_{c+j-1} + \Lambda) \leq n-1$ and the last inclusion. ■

Definition 2.10. Let X be a smooth projective variety of dimension m in $\mathbb{P}^n$. Then the morphism to the Grassmannian

$$\gamma_X : X \to \mathrm{Gr}\left(m, \mathbb{P}^n\right), \quad p \mapsto T_pX.$$

that takes point p on X to its tangent space T_pX is called the *Gauss map*.

Remark 2.3. Since X is a smooth projective variety, the Gauss map is a well-defined morphism. If X is given by polynomial equations, then we compute γ_X by mapping p to the kernel of the Jacobian matrix of the defining equations at p. If X is given by a parametrization, then we can represent the Gauss map γ_X by taking the derivatives of the parametrization. Furthermore, we can express the morphism in terms of Plücker coordinates.

Example 2.10. Consider the twisted cubic curve

$$X = \mathbb{V}(x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3).$$

Let $p = (p_0 : p_1 : p_2 : p_3) \in X$ then the tangent space of X at the point p is defined as $\mathbb{P}(\ker \mathcal{J}_X(p))$. The Jacobian of X at the point p is given by

$$\mathcal{J}_X = \begin{pmatrix} -p_2 & 2p_1 & -p_0 & 0 \\ -p_3 & p_2 & p_1 & -p_0 \\ 0 & -p_3 & 2p_2 & -p_1 \end{pmatrix},$$

the kernel of this matrix is 2 dimensional vector space in $\mathbb{A}^4$ therefore gives a line in projective space $\mathbb{P}^3$. Let $v_1 = (3p_0, 2p_1, p_2, 0)^T$ and $v_2 = (0, p_1, 2p_2, 3p_3)^T$. They are clearly linearly independent due to appearance of 0 in different places. Furthermore, note that since $p \in X$ we have

$$\mathcal{J}_X(p)v_1 = \begin{pmatrix} 4p_1^2 - 4p_0p_2 \\ 3p_1p_2 - 3p_0p_3 \\ 2p_2^2 - 2p_1p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathcal{J}_X(p)v_2 = \begin{pmatrix} 2p_1^2 - 2p_0p_2 \\ 3p_1p_2 - 3p_0p_3 \\ -4p_1p_3 + 4p_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So given $p \in X$ the tangent line T_pX is given by projective span $v_1 + v_2$. Therefore the Gauss map

$$\gamma_X : X \to Gr(1, \mathbb{P}^3)$$
$$p = (p_0 : p_1 : p_2 : p_3) \mapsto \text{row} \begin{pmatrix} 3p_0 & 2p_1 & p_2 & 0 \\ 0 & p_1 & 2p_2 & 3p_3 \end{pmatrix}$$

Recall that we can represent $Gr(1, \mathbb{P}^3) \subseteq \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ in terms of 2×2 minors of the matrix. Therefore the plücker coordinates of the tangent line T_pX, at the point p is

$$x_{12} = 3p_0p_1,\ x_{13} = 6p_0p_2,\ x_{14} = 9p_0p_3,\ x_{23} = 3p_1p_2,\ x_{24} = 6p_1p_3,\ x_{34} = 3p_2p_3.$$

Therefore we can write the morphism $\gamma_X : X \to Gr(1, \mathbb{P}^3)$ as

$$\gamma_X : (p_0 : p_1 : p_2 : p_3) \mapsto (p_0p_1 : 2p_0p_2 : 3p_0p_3 : p_1p_2 : 2p_1p_3 : p_2p_3).$$

For the question of how we found the basis v_1 and v_2 for the Jacobian $\mathcal{J}_X(p)$, we used the representation of twisted cubic curve in terms of parametrization given in the next example.

$\triangle$

Example 2.11. [10, Example 4.6] We are going to compute the Gauss map for the twisted cubic curve X again, using the description given in terms of its parametrition

$$v : \mathbb{P}^1 \to \mathbb{P}^3, \quad (s:t) \mapsto \left(s^3 : s^2t : st^2 : t^3\right).$$

Let $p = v(s:t) = (s^3 : s^2t : st^2 : t^3)$ then the tangent space T_pX is given by the row span of the matrix

$$\begin{pmatrix} 3s^2 & 2st & t^2 & 0 \\ 0 & s^2 & 2st & 3t^2 \end{pmatrix}$$

whose first row is the partial derivative of v with respect to s and second row is the partial derivative of v with respect to t. Note that writing $p = (p_0 : p_1 : p_2 : p_3) = (s^3 : s^2t : st^2 : t^3)$ we see that

$$(3p_0 : 2p_1 : p_2 : 0) = (3s^3 : 2s^2t : st^2 : 0) = (3s^2 : 2st : t^2 : 0)$$

and similarly

$$(0 : p_1 : 2p_2 : 3p_3) = (0 : s^2t : 2st^2 : 3t^3) = (0 : s^2 : 2st : 3t^2)$$

which was the basis of the tangent space at the point p in the previous example.

Now using Plücker coordinates for $Gr(1, \mathbb{P}^3) = \mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ we need to take 2×2 minors of the matrix

$$\begin{pmatrix} 3s^2 & 2st & t^2 & 0 \\ 0 & s^2 & 2st & 3t^2 \end{pmatrix},$$

which is

$$x_{12} = 3s^4,\ x_{13} = 6s^3t,\ x_{14} = 9s^2t^2,\ x_{23} = 3s^2t^2,\ x_{24} = 6st^3,\ x_{34} = 3t^4.$$

Therefore the morphism γ can be seen as a morphism through $\mathbb{P}^1 \to \mathbb{P}^5$ which is

$$(s:t) \mapsto (s^4 : 2s^3t : 3s^2t^2 : s^2t^2 : 2st^3 : t^4).$$

Note that after writing $p = (p_0 : p_1 : p_2 : p_3) = (s^3 : s^2t : st^2 : t^3)$, we get

$$(p_0p_1 : 2p_0p_2 : 3p_0p_3 : p_1p_2 : 2p_1p_3 : p_2p_3) = (s^5t : 2s^4t^2 : 3s^3t^3 : s^3t^3 : 2s^2t^4 : st^5)$$

and factoring out st we get exactly the same map with the previous example. $\triangle$

Theorem 2.11. *[10, Proposition 4.8] Let X be a projective variety in $\mathbb{P}^n$ of dimension m. For any projective subspace V of $\mathbb{P}^n$, the polar variety $P(X, V)$ is the pullback of the special Schubert variety via the Gauss map, that is*

$$P(X,V) = \gamma^{-1}(\Sigma_m(V))$$

Proof. We have seen that the condition that $V+p$ intersects X at $p \in X$ non-transversely is equivalent to $T_pX \in \Sigma_m(V)$. Therefore, the set of points in X where $V+p$ intersects X at $p \in X$ non-transversely is precisely the points $p \in X$ where the tangent plane T_pX is intersecting V in an unexpected way, hence $\gamma_X(p) \in \Sigma_m(V)$. Furthermore, since $\Sigma_m(V) = \Sigma_{\mathbf{a}}$ for some flag containing V, and since $\Sigma_{\mathbf{a}}$ is a closed subset of the Grassmannian $\mathrm{Gr}(m, \mathbb{P}^n)$, its pullback is a closed subvariety of X. ∎

Example 2.12. Consider the example of twisted cubic curve $X \subseteq \mathbb{P}^3$. We are going to calculate first polar variety of $P(X,V)$ for the line V given by row span of matrix

$$\begin{pmatrix} 4 & 0 & -1 & 0 \\ 4 & 0 & -1 & 1 \end{pmatrix}.$$

First we write the defining equation for special Schubert variety $\Sigma_1(V)$ in terms of Plücker coordinates.

Let L be another line given written as projective sum of 2 points $a = (a_0 : a_1 : a_2 : a_3)$ and $b = (b_0 : b_1 : b_2 : b_3)$. Write the Plücker coordinate of $L = (l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34})$, where l_{ij} is the determinant of the 2×2 submatrix formed by columns i and j of

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

The lines V and L intersect if and only if they share a point, which occurs if and only if their span is a three dimensional subspace of the affine cone $\mathbb{A}^4$. This can be calculated by the determinant

$$\det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ 4 & 0 & -1 & 0 \\ 4 & 0 & -1 & 1 \end{pmatrix} = 4a_1b_2 - 4a_2b_1 - a_0b_1 + a_1b_0 = 4\left(a_1b_2 - a_2b_1\right) - \left(a_0b_1 - a_1b_0\right) = 4l_{23} - l_{12}.$$

Therefore, the Plücker coordinates of the special Schubert variety is

$$\Sigma_1(V) = \mathbb{V}(-x_{12} + 4x_{23}) \subseteq \mathbb{P}^5. \tag{7}$$

We have shown in the Theorem 2.11that $P(X,V) = \gamma_X^{-1}(\Sigma_1(V))$. Also we have shown the equation of Gauss map γ_X for the twisted cubic curve in Example 2.10 and 2.11. We substitute $x_{12} = s^4$ and $x_{23} = s^2t^2$ to (7) to get

$$-s^4 + 4s^2t^2 = 0.$$

After factoring we see that the solutions that satisfies this in $\mathbb{P}^1$ are $(s:t) = (0:1)$ with multiplicity 2, as well as $(s:t)=(2:1)$ and $(s:t)=(-2:1)$. Mapping these parameter values back to $\mathbb{P}^3$ via our parameterization to twisted cubic $v(s:t)$, we get the points:

$$\begin{aligned} p_1 &= v(0:1) = (0:0:0:1) \\ p_2 &= v(2:1) = (8:4:2:1) \\ p_3 &= v(-2:1) = (-8:4:-2:1). \end{aligned}$$

In order to visualize this, we choose the hyperplane at infinity to be $H_\infty = \mathbb{V}(x_0)$. Therefore in the affine chart the first polar variety of the twisted cubic curve has two points

$$p_2 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right) \text{ and } p_3 = \left(-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}\right)$$

on the affine part of the twisted cubic curve as can be seen in the figure below.

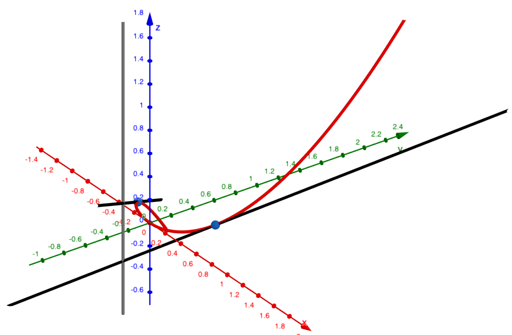

Figure 11: Twisted cubic curve X and its first polar variety

Here in the Figure 11, axis x, y, z denotes the x_1, x_2, x_3 respectively and it is twisted cubic curve in the affine chart. The red curve is the twisted cubic curve, the two blue points are the elements of the polar variety $P(X, V)$, where the V is the grey line. Note that the tangent spaces which are lines colored in black are passing through the V. We have also said that there is another point with multiplicity 2 which lies in the point at infinity, they correspond to directions of the twisted cubic curve in the z axis. $\triangle$

Corollary 2.12. *[10] An inclusion of projective linear subspaces gives a reverse inclusion of polar varieties. To be precise, for any fixed variety X in $\mathbb{P}^n$ and subspaces $V \subset V'$, we have $P(X, V') \subset P(X, V)$.*

Proof. Let $V \subset V'$ and let $\mathcal{V}$ be a flag containing both V and V'. In the Lemma 2.10 we have shown that $\Sigma_m(V) = \Sigma_{\mathbf{a}}$ and $\Sigma_m(V') = \Sigma_{\mathbf{a}'}$ for some sequences $\mathbf{a}$ and $\mathbf{a}'$. Since $V \subset V'$ we have

$\mathbf{a} \subset \mathbf{a}'$ as well. Using [19, Chapter 4.1] we have $\Sigma_{\mathbf{a}'} \subset \Sigma_{\mathbf{a}}$. Therefore after taking pullback along the Gauss map, we see that $P(X, V') \subset P(X, V)$. ■

Corollary 2.13 (Codimension of the i-th polar variety)**.** *Let X be a smooth projective variety of dimension m in $\mathbb{P}^n$, and let V be a generic linear subspace of dimension* $\dim V = c+i-2$ *where* $c = \operatorname{codim} X$. *Then the codimension of the i-th polar variety $P_i(X, V)$ in X is* $\operatorname{codim}_X(P_i(X, V)) = i$.

Proof. Fix a general projective subspace $L \subseteq \mathbb{P}^n$ of dimension $c + i - 2$ and a complete flag $\mathcal{L}$ containing L. We have a morphism of varieties $\gamma_X : X \to \mathrm{Gr}(m, \mathbb{P}^n)$ and by Theorem 2.11, we know that $P_i(X, L) = \gamma_X^{-1}(\Sigma_m(L))$. Furthermore, by Lemma 2.10

$$\Sigma_m(L) = \Sigma_{\mathbf{a}}, \qquad \text{where } \mathbf{a} = (\underbrace{1, \dots, 1}_{\text{i times}}, 0, \dots, 0).$$

By [19, Chapter 4.1], we have $\operatorname{codim}_{Gr(m,\mathbb{P}^n)}(\Sigma_{\mathbf{a}}) = \sum_{j=0}^{m} a_j = i$. Let $PGL_{n+1}(\mathbb{C})$ be projective general linear group, which is the quotient of general linear group $GL_{n+1}(\mathbb{C})$ by subgroup of scalar multiples of the identity matrix. Therefore it acts transitively on $Gr(m, \mathbb{P}^n)$. Furthermore, take a general element $M \in PGL_{n+1}(\mathbb{C})$, it also acts on the flag $\mathcal{L}$, which we denote as $\mathcal{L} \cdot M$. Therefore it induces an action on the subvariety $\Sigma_{\mathbf{a}}$ of $\mathrm{Gr}(m, \mathbb{P}^n)$, by acting on the flag $\mathcal{L}$.

Since M is general, by Kleiman's theorem in characteristic 0 [19, Theorem 1.7], applied to the transitive action of $PGL_{n+1}(\mathbb{C})$ on $Gr(m, \mathbb{P}^n)$, we have $\operatorname{codim}_X(P_i(X, L \cdot M)) = \operatorname{codim}_{Gr(m,\mathbb{P}^n)}(\Sigma_{\mathbf{a}}) = i$. Since M is a general element of $PGL_{n+1}(\mathbb{C})$, the translate $V := L \cdot M$ is also a general projective subspace of dimension $c + i - 2$. Therefore $\operatorname{codim}_X(P_i(X, V)) = i$. ■

2.4 Classical Polar Varieties in terms of Conormal Varieties

In this subsection, we are going to give another equivalent way of writing polar varieties using duality in projective space. We start by defining the dual of the projective space. The proof will be the imitation of the proof of [15, Lemma 3.24].

Definition 2.11. Define the dual of a projective space $(\mathbb{P}^n)^\vee$ to be set whose elements $H \in (\mathbb{P}^n)^\vee$ are the hyperplanes in $\mathbb{P}^n$.

Remark 2.4. We have already remarked that $(\mathbb{P}^n)^\vee$ is equal to the Grassmannian $Gr(n-1, \mathbb{P}^n)$ in Example 2.7, therefore this can be thought as projective space again.

Definition 2.12 (Conormal Variety)**.** Let X be a smooth projective variety. Conormal variety $CN(X)$ of X is the set

$$CN(X) = \{(p, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid p \in X, H \subseteq T_p X\} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee$$

Definition 2.13. The projection of $CN(X)$ onto the second coordinate is called the dual variety $X^\vee$. Its points are the hyperplanes in $\mathbb{P}^n$ that are tangent to X.

Remark 2.5. Since the projective space $\mathbb{P}^n$ is complete the projection of a closed set must be closed. Therefore $X^\vee$ must be closed subset of $(\mathbb{P}^n)^\vee$.

Example 2.13. By definition of conormal variety we have

$$CN(V) = \left\{(p, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid p \in V, T_pV \subseteq H\right\} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee .$$

Since V is already a projective subspace $T_pV = V$, therefore when projected to the second coordinate we see that

$$V^\vee = \left\{H \in (\mathbb{P}^n)^\vee \mid V \subseteq H\right\}.$$

Which means it is the set of hyperplanes in $\mathbb{P}^n$ that contain V. △

Theorem 2.14. *Let X be a smooth projective variety, and V be a subspace in $\mathbb{P}^n$. Denote the projection to first and second coordinates of the conormal variety to be π_1 and π_2. That is $\pi_1 : CN(X) \to X$ and $\pi_2 : CN(X) \to X^\vee$. Then*

$$P(X, V) = \pi_1(\pi_2^{-1}(V^\vee))$$

Proof. Recall that for a subspace V its dual is $V^\vee$ is the set of hyperplanes H containing L, that is

$$V^\vee = \{H \in (\mathbb{P}^n)^\vee \mid V \subset H\}$$

We first describe the inverse image of $V^\vee$ under the second projection $\pi_2 : CN(X) \to X^\vee$,

$$\begin{aligned}
\pi_2^{-1}(V^\vee) &= \{(p, H) \in CN(X) \mid H \in V^\vee\} \\
&= \{(p, H) \mid p \in X, V \subset H, T_pX \subset H\} \\
&= \{(p, H) \mid p \in X, V + T_pX \subset H\}.
\end{aligned}$$

Now we consider the image of this set under the first projection $\pi_1 : CN(X) \to X$

$$\begin{aligned}
\pi_1(\pi_2^{-1}(V^\vee)) &= \{p \in X \mid \exists H \in (\mathbb{P}^n)^\vee, V + T_pX \subset H\} \\
&= \{p \in X \mid \dim(V + T_pX) < n\} \\
&= \{p \in X \mid (V + p) \not\pitchfork T_pX\} \\
&= P(X, V)
\end{aligned}$$

hence, we are done. ■

This point of view is going to allow us to show that multidegree which we define afterwards in Chapter 5 is related to the degree of the polar variety.

2.5 Computational Perspectives of Classical Polar Varieties

In this subsection we give two computational approaches of defining classical polar varieties using rank conditions of matrices. The first approach is the way polar varieties are defined in the paper [32]. The second approach, however, is the way polar varieties are defined in the paper [5]. Our goal for each of the characterizations is to write the non-transversal intersection condition in terms of rank conditions of matrices. The key difference between the two descriptions is on how V is given.

Let $V = \mathbb{P}(\hat{V})$ be a projective subspace of dimension k. We are going to use two notations for the rest of the chapter. We write the full rank matrix $M_V \in \mathbb{C}^{(n+1)\times(k+1)}$ to write $\hat{V} = \operatorname{col} M_V$ so that, we can write $V = \mathbb{P}(\operatorname{col}(M_V))$. We write the full rank matrix $N_V \in \mathbb{C}^{(n-k)\times(n+1)}$ to write $\hat{V} = \ker(N_V)$, so that, we can write $V = \mathbb{P}(\ker(N_V))$. In other words, we have a short exact sequence

$$0 \to \mathbb{C}^{k+1} \xrightarrow{M_V} \mathbb{C}^{n+1} \xrightarrow{N_V} \mathbb{C}^{n-k} \to 0.$$

So depending on how V is given, we have two different ways to calculate the polar variety. For the upcoming theorem, let $V = \mathbb{P}(M_V)$ and $X = \mathbb{V}(F_1, \ldots, F_s) \subseteq \mathbb{P}^n$ smooth, with codimension c as before. Recall that we denote the Jacobian of X by $\mathcal{J}_X$.

Theorem 2.15. *Let $p \in X$, then $V + p$ intersects X at p non-transversely if and only if $\operatorname{rk}(\mathcal{J}_X(p) \cdot M_V) < c$, where $\mathcal{J}_X(p) \cdot M_V$ is matrix multiplication and Jacobian $\mathcal{J}_X(p)$ is evaluated at representative of p.*

Proof. We have shown in the remark 2.1 that $V+p$ intersects X at p non-transversely is equivalent to

$$\dim(V \cap T_pX) > k - c. \tag{8}$$

where $\dim V = k$ and $\operatorname{codim} X = c$.

Now since $V = \mathbb{P}(\operatorname{col} M_V)$ and $T_pX = \mathbb{P}(\ker \mathcal{J}_X(p))$, using lemma 2.2 for the intersections, we get

$$V \cap T_pX = \mathbb{P}(\operatorname{col} M_V) \cap \mathbb{P}(\ker \mathcal{J}_X(p)) = \mathbb{P}(\operatorname{col} M_V \cap \ker \mathcal{J}_X(p));$$

therefore, on dimensions we have

$$\dim(V \cap T_pX) = \dim(\mathbb{P}(\operatorname{col} M_V \cap \ker \mathcal{J}_X(p))) = \dim(\operatorname{col} M_V \cap \ker \mathcal{J}_X(p)) - 1. \tag{9}$$

Now using lemma 2.7 on $\mathcal{J}_X(p) \cdot M_V : \mathbb{C}^{k+1} \to \mathbb{C}^{n+1} \to \mathbb{C}^s$ we have the following equality

$$\dim(\ker(\mathcal{J}_X(p) \cdot M_V)) = \dim(\ker M_V) + \dim(\operatorname{col} M_V \cap \ker \mathcal{J}_X(p)). \tag{10}$$

For the equation (10), we use rank nullity on $\dim \ker(\mathcal{J}_X(p) \cdot M_V)$, write $\dim(\ker M_V) = 0$ since

M_V is full column rank, and we use (9) on $\dim(\operatorname{col} M_V \cap \ker \mathcal{J}_X(p))$ to get

$$(k+1) - \operatorname{rk}(\mathcal{J}_X(p) \cdot M_V) = \dim(V \cap T_p X) + 1.$$

Hence by rearranging and using the inequality (8), we obtain

$$\operatorname{rk}(\mathcal{J}_X(p) \cdot M_V) = (k+1) - (\dim(V \cap T_p X) + 1) = k - \dim(V \cap T_p X) < k + (c-k) = c$$

concluding the first direction.

Now for the converse direction assume that $\operatorname{rk}(\mathcal{J}_X(p) \cdot M_V) < c$, and we would like to show that $n > \dim((V+p) + T_p X)$. By remark 2.1 it is enough to show $\dim(V \cap T_p X) > k - c$. Now similarly to the converse case, using lemma 2.7 we get

$$\dim(\ker(\mathcal{J}_X(p) \cdot M_V)) = \dim(\ker M_V) + \dim(\operatorname{col} M_V \cap \ker \mathcal{J}_X(p)).$$

Afterwards, we use rank nullity theorem on $\dim(\ker(\mathcal{J}_X(p) \cdot M_V))$, write $\dim(\ker M_V) = 0$, and use (9) to get

$$(k+1) - \operatorname{rk}(\mathcal{J}_X(p) \cdot M_V) = \dim(V \cap T_p X) + 1,$$

thus by rearranging

$$\dim(V \cap T_p X) = k - \operatorname{rk}(\mathcal{J}_X(p) \cdot M_V) > k - c.$$

■

Corollary 2.16. *Let X be a smooth projective variety in $\mathbb{P}^n$ given by $\mathbb{V}(F_1, \ldots, F_s)$. Let V be a projective subspace of $\mathbb{P}^n$ given by $V = \mathbb{P}(\operatorname{col} M_V)$. Let Δ be the ideal*

$$\Delta = \langle F_1, \ldots, F_s \rangle + \langle c \times c \textit{ minors } \mathcal{J}_X M_V \rangle .$$

Then the polar variety $P(X, V)$ of X with respect to subspace V is equal to $\mathbb{V}(\Delta)$.

Example 2.14. Let us revisit the ellipsoid example $X = \mathbb{V}(F) \subseteq \mathbb{P}^3$ again which was given in Example 2.4, where $F = c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - x_0^2$. The dimension of X is 2, therefore the codimension is $c = 1$. The Jacobian of X is 1×4 matrix

$$\mathcal{J}_X = \begin{pmatrix} -2x_0 & 2c_1x_1 & 2c_2x_2 & 2c_3x_3 \end{pmatrix}$$

Let V be the projective line spanned by the affine point $v = [v_0 : v_1 : v_2 : v_3]$ and $w = [w_0 : w_1 :$

$w_2 : w_3]$. We represent V as the projectivization of the column span of 4×2 matrix M_V:

$$M_V = \begin{pmatrix} v_0 & w_0 \\ v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \end{pmatrix}.$$

We use the Corollary 2.16 to write the ideal of $P(X, V)$, we first compute the product $\mathcal{J}_X M_V$

$$\mathcal{J}_X M_V = \begin{pmatrix} -2v_0x_0 + 2c_1v_1x_1 + 2c_2v_2x_2 + 2c_3v_3x_3 & -2w_0x_0 + 2c_1w_1x_1 + 2c_2w_2x_2 + 2c_3w_3x_3 \end{pmatrix}.$$

Because the codimension is $c = 1$, the unsaturated ideal Δ is

$$\Delta = \left\langle c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - x_0^2, -v_0x_0 + c_1v_1x_1 + c_2v_2x_2 + c_3v_3x_3, -w_0x_0 + c_1w_1x_1 + c_2w_2x_2 + c_3w_3x_3 \right\rangle.$$

Therefore the system of equation defining the polar variety is

$$\begin{aligned} -v_0x_0 + c_1v_1x_1 + c_2v_2x_2 + c_3v_3x_3 &= 0 \\ -w_0x_0 + c_1w_1x_1 + c_2w_2x_2 + c_3w_3x_3 &= 0 \\ c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - x_0^2 &= 0, \end{aligned}$$

which is what we have found using geometric arguments in the Example 2.4. $\triangle$

Example 2.15 (Twisted Cubic Example)**.** To observe the polar variety for projective varieties having codimension greater than 1, we use twisted cubic curve $X \subseteq \mathbb{P}^3$ like in the Example 2.12. X is defined by ideal generated by three quadrics $F_1 = x_1^2 - x_0x_2$, $F_2 = x_1x_2 - x_0x_3$ and $F_3 = x_2^2 - x_1x_3$. The dimension of X is 1 therefore the codimension of X is $c = 2$.

The Jacobian matrix $\mathcal{J}_X$ is 3×4 matrix

$$\mathcal{J}_X = \begin{pmatrix} -x_2 & 2x_1 & -x_0 & 0 \\ -x_3 & x_2 & x_1 & -x_0 \\ 0 & -x_3 & 2x_2 & -x_1 \end{pmatrix}.$$

Consider the first polar variety $i = 1$. The dimension of V is $\dim V = c+i-2 = 1$ which means V is a line. Define V as the line spanned by the points $v = [4 : 0 : -1 : 0]$ and $w = [4 : 0 : -1 : 1]$. The column span matrix M_V is

$$M_V = \begin{pmatrix} 4 & 4 \\ 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

When we multiply $\mathcal{J}_X M_V$ we obtain 3×2 matrix

$$\mathcal{J}_X M_V = \begin{pmatrix} -x_2 & 2x_1 & -x_0 & 0 \\ -x_3 & x_2 & x_1 & -x_0 \\ 0 & -x_3 & 2x_2 & -x_1 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4x_2 + x_0 & -4x_2 + x_0 \\ -4x_3 - x_1 & -4x_3 - x_1 - x_0 \\ -2x_2 & -2x_2 - x_1 \end{pmatrix}$$

Since the codimension is $c = 2$, we calculate 2×2 minors of this product matrix, which are

$$\begin{aligned} m_{12} &= 4x_0x_2 - x_0^2 = x_0(4x_2 - x_0) \\ m_{13} &= 4x_1x_2 - x_0x_1 = x_1(4x_2 - x_0) \\ m_{23} &= 4x_1x_3 + x_1^2 - 2x_0x_2. \end{aligned}$$

Therefore the unsaturated ideal Δ is generated by

$$\Delta = \langle F_1, F_2, F_3, m_{12}, m_{13}, m_{23} \rangle .$$

Note that since $F_1 = x_1^2 - x_0x_2$ we can substitute $x_0x_2 = x_1^2$ to m_{23} to get

$$m_{23} = 4x_1x_3 - x_1^2 = x_1(4x_3 - x_1),$$

to have similar form with m_{12} and m_{13}. Now if we solve the system Δ symbolically using OSCAR [26] we get the minimal primes of Δ to be

$$\begin{aligned} \langle x_2 + 2x_3, x_1 - 4x_3, x_0 + 8x_3 \rangle \\ \langle x_2 - 2x_3, x_1 - 4x_3, x_0 - 8x_3 \rangle \\ \langle x_2, x_1, x_0 \rangle \end{aligned}$$

therefore it corresponds to points in projective space $\mathbb{P}^3$

$$\left\{ \left[1 : -\frac{1}{2} : \frac{1}{4} : -\frac{1}{8}\right], \left[1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8}\right], [0 : 0 : 0 : 1] \right\}.$$

These are precisely the three points we have seen in the Example 2.12 with the Figure 11. $\triangle$

If $V = \mathbb{P}(\ker N_V)$ for some matrix $N_V \in \mathbb{C}^{(n-c-i+2)\times(n+1)}$, then we use the lemma below:

Lemma 2.17. *Let $i \in \{0, \dots, \dim X\}$, suppose we have two full rank matrices $M_V \in \mathbb{C}^{(n+1)\times(c+i-1)}$ and $N_V \in \mathbb{C}^{(n-c-i+2)\times(n+1)}$ such that $\ker N_V = \operatorname{col} M_V$. Then for any $p \in X$ we have*

$$\operatorname{rk}(\mathcal{J}_X(p) M_V) < c$$

if and only if

$$\operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} < n-i+2.$$

Proof. We start by assuming $\operatorname{rk}(\mathcal{J}_X(p)M_V) < c$. Now we use the rank-nullity Theorem 2.6 on the linear transformation given by a matrix

$$\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} : \mathbb{C}^{n+1} \to \mathbb{C}^{s+n-c-i+2},$$

then through the observations $\ker\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} = \ker \mathcal{J}_X(p) \cap \ker N_V$, assumption $\operatorname{col} M_V = \ker N_V$, and by the equation (1), we get following chain of equalities

$$\begin{aligned}
n+1 &= \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} + \dim\ker\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix}.\\
&= \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} + \dim(\ker(\mathcal{J}_X(p)) \cap \ker N_V)\\
&= \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} + \dim(\ker \mathcal{J}_X(p) \cap \operatorname{col} M_V)\\
&= \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} + \dim(\operatorname{col} M_V) - \dim\operatorname{col}(\mathcal{J}_X(p)M_V)\\
&= \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} + (c+i-1) - \dim\operatorname{col}(\mathcal{J}_X(p)M_V)\\
&> \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} + i - 1
\end{aligned}$$

hence

$$n-i+2 > \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix}$$

proving the first implication.

For the converse assume that $\operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} < n-i+2$. Using rank-nullity Theorem 2.6 we have

$$-\dim\left(\ker\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix}\right) = -(n+1) + \operatorname{rk}\begin{pmatrix}\mathcal{J}_X(p)\\ N_V\end{pmatrix} < -n-1+(n-i+2) = -i+1.$$

Therefore after using this inequality in (1)

$$\begin{aligned}
\operatorname{rk}(\mathcal{J}_X(p)M_V) &= \operatorname{rk} M_V - \dim(\ker \mathcal{J}_X(p) \cap \operatorname{Im} M_V) \\
&= \operatorname{rk} M_V - \dim(\ker \mathcal{J}_X(p) \cap \ker N_V) \\
&= \operatorname{rk} M_V - \dim\left(\ker \begin{pmatrix} \mathcal{J}_X(p) \\ N_V \end{pmatrix}\right) \\
&< (c+i-1) + (-i+1) = c
\end{aligned}$$

we proved the last implication. ■

Corollary 2.18. *Define the ideal D to be*

$$D = \langle F_1, \dots, F_s \rangle + \left\langle (n-i+2) \times (n-i+2) \text{ minors of } \begin{pmatrix} \mathcal{J}_X \\ N_V \end{pmatrix} \right\rangle$$

where N_V is $(n-c-i+2) \times (n+1)$ matrix. where $V = \mathbb{P}(\ker N_V)$ then the polar variety $P(X, V)$ is $\mathbb{V}(D)$.

This point of view of the polar variety is going to allow us to see the degree of the polar varieties as algebraic degree of a particular optimization problem. The details will be given in the chapter 5.5.

Example 2.16. To demonstrate this computational approach, we recalculate the polar variety of the ellipsoid $X = \mathbb{V}(c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - x_0^2) \subset \mathbb{P}^3$ using the kernel formulation. As before $\dim X = 2$ therefore codimension is $c = 1$. The Jacobian of X is 1×4 matrix

$$\mathcal{J}_X = \begin{pmatrix} -2x_0 & 2c_1x_1 & 2c_2x_2 & 2c_3x_3 \end{pmatrix}.$$

We compute the $i = 2$ polar variety. Again $\dim V = c + i - 2 = 1$ therefore V is a line. We use the same projective subspace V that passes through $(2, 0, 0)$ and $(0, 2, 0)$ in the affine chart. However this time rather than defining V via its basis points we are going to define it as an intersection of two hyperplanes $-2x_0 + x_1 + x_2 = 0$ and $x_3 = 0$. Therefore, $V = \mathbb{V}(\ker N_V)$ where N_V is a matrix of size $(n-c-i+2) \times (n+1) = 2 \times 4$ matrix

$$N_V = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The stacked matrix is

$$\begin{pmatrix} \mathcal{J}_X \\ N_V \end{pmatrix} = \begin{pmatrix} -2x_0 & 2c_1x_1 & 2c_2x_2 & 2c_3x_3 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We want the rank of this stacked matrix to be strictly less than $n - i + 2 = 3$. Therefore, we calculate the 3×3 minors of this 3×4 matrix. Due to the last row of the matrix, this is equivalent to checking 2×2 minors of the submatrix

$$\begin{pmatrix} -2x_0 & 2c_1x_1 & 2c_2x_2 \\ -2 & 1 & 1 \end{pmatrix}.$$

Calculating the three 2×2 minors, we get

$$\begin{aligned} m_{12} &= (-2x_0)(1) - (2c_1x_1)(-2) = -2x_0 + 4c_1x_1 \\ m_{13} &= (-2x_0)(1) - (2c_2x_2)(-2) = -2x_0 + 4c_2x_2 \\ m_{23} &= (2c_1x_1)(1) - (2c_2x_2)(1) = 2c_1x_1 - 2c_2x_2. \end{aligned}$$

Notice that $m_{23} = \frac{1}{2}m_{12} - \frac{1}{2}m_{13}$ and is therefore redundant in the ideal. Hence,

$$D = \langle c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - x_0^2,\ -2x_0 + 4c_1x_1,\ -2x_0 + 4c_2x_2\rangle$$

which corresponds to the same ideal, we have derived in the Example 2.4. △

Remark 2.6. The definition of polar variety given in [2] is given by the ideal D in Corollary 2.18. The only difference is the indexing. However, if we restrict ourselves to the affine case by dehomogenizing, we obtain the same varieties, as stated in the Lemma 2.19 below. It describes the ideal of the affine polar variety $P(X, V) \cap \mathbb{A}^n$ when the subspace V is contained in the hyperplane at infinity H_∞. This will be useful in the Section 5.5.

Lemma 2.19. *Let $X \subseteq \mathbb{P}^n$ be a projective variety given by homogeneous ideal $I_X = \langle F_1, \ldots, F_s\rangle$. Let the hyperplane at infinity be $H_\infty = \mathbb{V}(x_0)$ and affine chart $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$. Define the corresponding affine variety $Y = X \cap \mathbb{A}^n$ by the ideal $I_Y = \langle f_1, \ldots, f_s\rangle$ where $f_j(x_1, \ldots, x_n) = F_j(1, x_1, \ldots, x_n)$.*

Let $N_V \in \mathbb{C}^{(n-c-i+2)\times(n+1)}$ be a matrix of full row rank such that $V := \mathbb{P}(\ker N_V) \subseteq H_\infty$. The matrix N_V is row equivalent to a block matrix of the form

$$\begin{pmatrix} 1 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{(n-c-i+1)\times 1} & N' \end{pmatrix}$$

where $N' \in \mathbb{C}^{(n-c-i+1)\times n}$ a full rank matrix. Then affine polar variety $P(X, V) \cap \mathbb{A}^n$ is given by the ideal

$$\langle f_1, \ldots, f_s\rangle + \left\langle (n-i+1) \times (n-i+1) \text{ minors of } \begin{pmatrix} \mathcal{J}_Y \\ N' \end{pmatrix} \right\rangle. \tag{11}$$

Proof. First of all note that since $\mathbb{P}(\ker N_V) \subseteq H_\infty$, the coordinate vector $(1, 0, \ldots, 0)$ must lie in the row span of N_V, hence after enough row operations, N_V is row equivalent to

$$\begin{pmatrix} 1 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{(n-c-i+1)\times 1} & N' \end{pmatrix}$$

where $N' \in \mathbb{C}^{(n-c-i+1)\times n}$. Take any $y \in Y$, then we have

$$\begin{aligned} \operatorname{rk}\begin{pmatrix} \mathcal{J}_X(y) \\ N_V \end{pmatrix} &= \operatorname{rk}\begin{pmatrix} \mathcal{J}_0(y) & \mathcal{J}_Y(y) \\ 1 & \mathbf{0} \\ \mathbf{0} & N' \end{pmatrix} \\ &= \operatorname{rk}\begin{pmatrix} \mathbf{0} & \mathcal{J}_Y(y) \\ 1 & \mathbf{0} \\ \mathbf{0} & N' \end{pmatrix} \\ &= \operatorname{rk}\begin{pmatrix} \mathcal{J}_Y(y) \\ N' \end{pmatrix} + 1, \end{aligned}$$

where $\mathcal{J}_0(y)$ is the column vector of partial derivatives with respect to x_0. Recall that $P(X,V)$ is given by the ideal

$$\langle F_1, \ldots, F_s \rangle + \left\langle (n-i+2)\times(n-i+2) \text{ minors } \begin{pmatrix} \mathcal{J}_X \\ N_V \end{pmatrix} \right\rangle. \tag{12}$$

It therefore suffices to show that the variety defined by the ideal (12) restricted to $x_0 = 1$ coincides with the affine variety given by the ideal (11).

Now for a point $\hat{y} = (1, y_1, \ldots, y_n) \in \mathbb{A}^{n+1}$ in the affine cone of $\mathbb{P}^n$, satisfies $F_j(1, y_1, \ldots, y_n) = 0$ for all $j = 1, \ldots, s$ and the rank condition

$$\operatorname{rk}\begin{pmatrix} \mathcal{J}_X(\hat{y}) \\ N_V \end{pmatrix} < n - i + 2$$

if and only if it satisfies $f_j(y_1, \ldots, y_n) = 0$ for all $j = 1, \ldots, s$ and the rank condition

$$\operatorname{rk}\begin{pmatrix} \mathcal{J}_Y(\hat{y}) \\ N' \end{pmatrix} < n - i + 1.$$

The equivalence of the rank conditions follows from the identity

$$\operatorname{rk}\begin{pmatrix} \mathcal{J}_X(\hat{y}) \\ N_V \end{pmatrix} = \operatorname{rk}\begin{pmatrix} \mathcal{J}_Y(\hat{y}) \\ N' \end{pmatrix} + 1,$$

established above, and the equivalence of the polynomial conditions follows from the $f_j = F_j|_{x_0=1}$.

Therefore the ideal of $P(X, V) \cap \mathbb{A}^n$ is given by

$$\langle f_1, \ldots, f_s \rangle + \left\langle (n-i+1) \times (n-i+1) \text{ minors of } \begin{pmatrix} \mathcal{J}_Y \\ N' \end{pmatrix} \right\rangle,$$

and that concludes the proof. ■

3 Reciprocal Polar Varieties

Following the references [25, 28, 29], we introduce the reciprocal polar varieties. First we will introduce the notion of polarity to have an orthogonal complement in projective space. The remainder of the section is devoted to understanding the geometry of reciprocal polar varieties, which will be used in the applications.

3.1 Polarity

To define reciprocal polar varieties, we need to have a notion of orthogonal complement in projective space, which will be called the polar. The geometric intuition we want to keep in mind is illustrated in the example below.

Example 3.1. In this example, we will be working in the projective plane $\mathbb{P}^2$ whose coordinates will be given by $(x_0 : x_1 : x_2)$. To draw the pictures we will use the real part of the affine chart $\{x_0 = 1\}$.

Let $Q \subseteq \mathbb{P}^2$ be a conic defined as the zero set of the $q = x_1^2 + x_2^2 - x_0^2$, therefore in the affine chart it corresponds to a circle centered at zero of radius 1. Now take a point $p = (1 : 2 : 0)$, we are interested in the points in Q, whose tangent lines passes through p and take projective span of them. Since Q is a hypersurface this is directly the classical polar variety $P(Q, p)$. Since Q is a conic, the classical polar variety will be a set of two points, namely $(1 : \frac{1}{2} : \pm\frac{\sqrt{3}}{2})$. The line passes through these two points which will be denoted as $p^\perp = \mathbb{V}(-x_0 + 2x_1)$ is called the polar line of the point p.

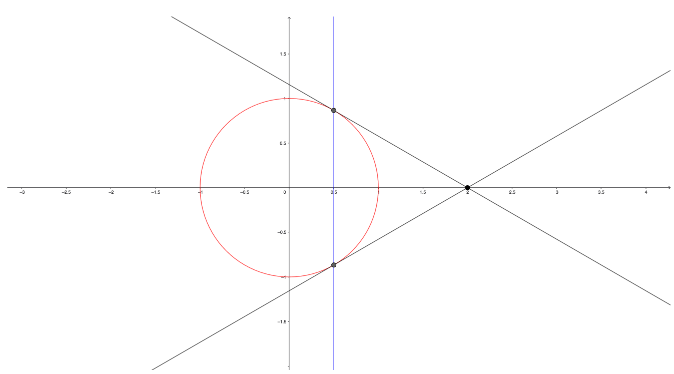

Figure 12: $p^\perp$ in the affine picture

In Figure 12, the red circle is conic Q, and blue line is $p^\perp$ in the affine chart $x_0 = 1$. The black point in $(2, 0)$ is the point p, and black lines on Q are showcasing how tangent lines intersect the point p.

Observe that if p was a point lying on the conic Q, $p^\perp$ would be equal to the T_pQ as the only point in Q whose tangent passes through p is the tangent line at p itself. This is a general statement that will be proved in Lemma 3.6.

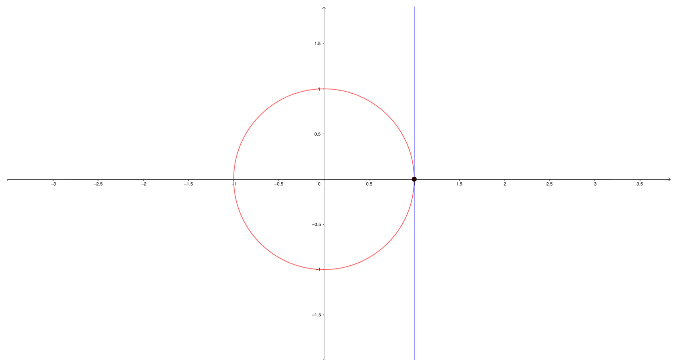

Figure 13: p lying on the conic Q

In the Figure 13, the black point $p \in Q$ now and the blue line which is polar $p^\perp$ is exactly T_pQ.

We would like a notion of orthogonal complement that would give us $p^{\perp\perp} = p$ as well. Since the dual of a point is a line, the dual of a line must be a point. Therefore, for a line $L \subseteq \mathbb{P}^2$ we define $L^\perp$ to be the point where the tangent lines of the points $L \cap Q$ meet. For a slightly different example, suppose the line L in $\mathbb{P}^2$ is given by the projective span of two points p_1 and p_2. Choose the points to be $p_1 = \left(1 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}}\right)$ and $p_2 = \left(1 : \frac{1}{\sqrt{2}} : -\frac{1}{\sqrt{2}}\right)$. Therefore the dual $L^\perp$ of the line L is given by the equation

$$L^\perp = \mathbb{V}\left(-x_0 + \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2\right) \cap \mathbb{V}\left(-x_0 + \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2\right) = \left\{\left(1 : \sqrt{2} : 0\right)\right\}.$$

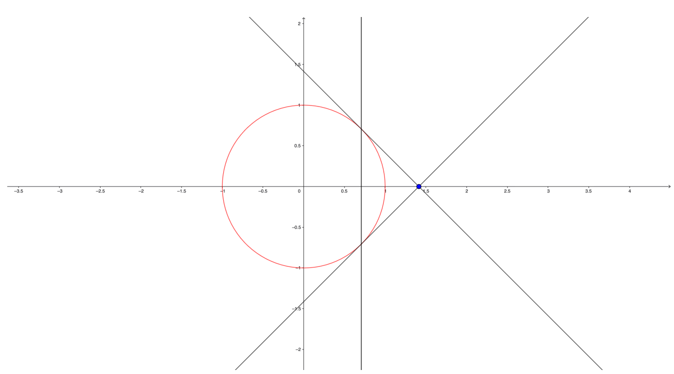

Figure 14: $L^\perp$ in the affine picture

In the Figure 14 note that the blue point is now the $(p_1+p_2)^{\perp}$ and the black line intersecting Q is the (p_1+p_2)

Note that the conic Q was given by the polynomial $q = x_1^2 + x_2^2 - x_0^2$ to have a nonempty real picture, but that does not have to be the case. For instance for Q_{ED} defined by the polynomial $q = x_1^2+x_2^2+x_0^2$ does not have real solutions in the affine picture. However the same picture would happen in complex setting. △

We use the notation for polarity in [31, Section 4]. Fix a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$ given by degree two polynomial $q \in \mathbf{k}[x_0, \dots, x_n]$. Write the degree two polynomial of the form $q(x) = x^T A x$, where A is a symmetric matrix of size $(n+1) \times (n+1)$, then $\nabla q(x) = 2Ax$. Note that Q is nonsingular is equivalent to $\det A \neq 0$.

Definition 3.1. The quadric Q given by the polynomial q induces a polarity map

$$\partial_q : \mathbb{P}^n \to (\mathbb{P}^n)^{\vee}.$$
$$p = (p_0 : \dots : p_n) \mapsto \left(\frac{\partial q}{\partial x_0}(p) : \dots : \frac{\partial q}{\partial x_n}(p) \right)$$

Let V be a projective subspace of $\mathbb{P}^n$, then we define the polar of V to be $V^{\perp}$ which is defined as

$$V^{\perp} := \partial_q(V)^{\vee} = \{y \in \mathbb{P}^n \mid \forall p \in V, \sum_{i=0}^{n} \frac{\partial q}{\partial x_i}(p) y_i = 0\}.$$

Example 3.2. Let $q \in \mathbf{k}[x_0, \dots, x_n]$ to be the $q := q_{ED} = \sum_{i=0}^n x_i^2$. Let $v = (v_0 : \dots : v_n) \in \mathbb{P}^n$, this is 0 dimensional subspace of $\mathbb{P}^n$, then its polar is by definition

$$v^{\perp} = \{y \in \mathbb{P}^n \mid v_0 y_0 + \dots + v_n y_n = 0\},$$

which is the orthogonal complement of the line v in $\mathbb{A}^{n+1}$. △

Proposition 3.1. *Consider the affine cone $\mathbb{A}^{n+1}$ of projective space $\mathbb{P}^n$, let $\hat{u}, \hat{v} \in \mathbb{A}^{n+1}$, and quadratic q, then the function defined as $\langle \hat{u}, \hat{v} \rangle_Q = \sum_{i=0}^n \frac{\partial q}{\partial x_i}(\hat{u}) \hat{v}_i$ is a symmetric bilinear form.*

Proof. We start by showing linearity in the first argument. Since q is a homogeneous degree two polynomial, its partial derivatives are homogeneous linear polynomials, therefore when considered as linear functionals they are linear.

For linearity in the second argument observe that this form is defined by multiplication and summation, therefore the map is linear in the second argument.

Now we are going to show the symmetric property. Since q is quadratic it is given by $q(\hat{u}) = \hat{u}^T A \hat{u}$ for some symmetric $(n+1) \times (n+1)$ matrix A. Furthermore, its gradient ∇q is given by matrix multiplication $\nabla q(\hat{u}) = 2A\hat{u}$. Using this representation we get

$$\langle \hat{u}, \hat{v} \rangle_Q = \nabla q(\hat{u}) \cdot \hat{v} = 2(A\hat{u}) \cdot \hat{v} = 2(A\hat{u})^T \hat{v} = 2\hat{u}^T A^T \hat{v} = 2\hat{u}^T A \hat{v}.$$

Now taking the transpose of the entire expression, since $\langle \hat{u}, \hat{v} \rangle_Q$ is just a scalar, we get

$$\langle \hat{u}, \hat{v} \rangle_Q = \langle \hat{u}, \hat{v} \rangle_Q^T = (2\hat{u}^T A \hat{v})^T = 2\hat{v}^T A^T \hat{u} = 2\hat{v}^T A \hat{u} = \langle \hat{v}, \hat{u} \rangle_Q$$

proving the symmetric property. ■

Remark 3.1. We note that although $\langle -, - \rangle_Q$ is a symmetric bilinear form, it may not be positive definite; hence, it is not necessarily an inner product, which is a symmetric bilinear form that is also positive definite. Therefore, it does not inherit all the properties of inner product. However, symmetric and bilinear alone give enough properties.

Definition 3.2. Let $\hat{V} \subseteq \mathbb{A}^{n+1}$ be a linear subspace; define the polar of $\hat{V}$ with respect to $\langle -, - \rangle_Q$ as the set $\{\hat{u} \in \mathbb{A}^{n+1} \mid \forall p \in \hat{V}, \langle u, p \rangle_Q = 0\}$ and it will be denoted as $\hat{V}^\perp$.

We note that the following Proposition 3.2, holds for any non-degenerate symmetric bilinear form, and a proof can be found in any linear algebra text on bilinear forms.

Proposition 3.2. *Let $\langle -, - \rangle_Q$ be a symmetric bilinear form in $\mathbb{A}^{n+1}$ then for any subspace $\hat{U}$ and $\hat{V}$ in $\mathbb{A}^{n+1}$, we have the following properties:*

1. *$\hat{U}^\perp$ is again a linear subspace of $\mathbb{A}^{n+1}$*
2. *$\hat{U}^{\perp\perp} = \hat{U}$*
3. *$\hat{U} \subseteq \hat{V}$ implies $\hat{V}^\perp \subseteq \hat{U}^\perp$*
4. *$\dim \hat{U} + \dim \hat{U}^\perp = n + 1$*

Remark 3.2. Since the elements of $\mathbb{P}^n$ depend on the choice of an affine representative, the evaluation on symmetric bilinear form $\langle -, - \rangle_Q$ is not well defined for the points in projective space $\mathbb{P}^n$. However, since $\langle -, - \rangle_Q$ is bilinear, the condition of polarity $\langle u, v \rangle_Q = 0$ is well defined for $u, v \in \mathbb{P}^n$. Therefore for $u, v \in \mathbb{P}^n$, we are going to write $\langle u, v \rangle_Q = 0$ to mean that their affine representatives are polar to each other in $\mathbb{A}^{n+1}$.

Proposition 3.3. *Let $\hat{V} \subseteq \mathbb{A}^{n+1}$ be a linear subspace, then we have*

$$\mathbb{P}(\hat{V})^\perp = \mathbb{P}\left(\hat{V}^\perp\right).$$

Proof. We are going to show $\mathbb{P}(\hat{V})^{\perp} \subseteq \mathbb{P}(\hat{V}^{\perp})$ first. Let $p \in \mathbb{P}(\hat{V})^{\perp}$ then for any $u \in \mathbb{P}(\hat{V})$ we have $\langle p, u \rangle_Q = 0$. Therefore for affine representative $\hat{p} \in \mathbb{A}^{n+1} \setminus \{0\}$ that is $[\hat{p}] = p$, and affine representative $\hat{u} \in \hat{V} \setminus \{0\}$, $[\hat{u}] = u$, we must have $\langle \hat{p}, \hat{u} \rangle_Q = 0$. We would like to show that $\hat{p} \in \hat{V}^{\perp}$. For any $\hat{u} \in \hat{V} \setminus \{0\}$, $\langle \hat{p}, \hat{u} \rangle_Q = 0$ already holds, for $\hat{u} = 0$, $\langle \hat{p}, \hat{u} \rangle_Q = 0$ also holds. Therefore we have $\langle \hat{p}, \hat{u} \rangle_Q = 0$ for all $\hat{u} \in \hat{V}$, which means $\hat{p} \in \hat{V}^{\perp}$. Hence $p = [\hat{p}] \in \mathbb{P}(\hat{V}^{\perp})$.

For the converse let $p \in \mathbb{P}(\hat{V}^{\perp})$ therefore there is $\hat{p} \in \hat{V}^{\perp}$ such that $[\hat{p}] = p$. Let $u \in \mathbb{P}(\hat{V})$ and affine representative $\hat{u} \in \hat{V} \setminus \{0\}$ such that $[\hat{u}] = u$. Since $\hat{p} \in \hat{V}^{\perp}$ it must be the case that $\langle \hat{u}, \hat{p} \rangle_Q = 0$, hence $\langle u, p \rangle_Q = 0$, therefore $p \in \mathbb{P}(\hat{V})^{\perp}$ ■

Corollary 3.4. *Let U and V be projective subspaces of $\mathbb{P}^n$, we have the following properties:*

1. *$V^{\perp}$ is again a projective subspace of $\mathbb{P}^n$*
2. *$V^{\perp\perp} = V$*
3. *$U \subseteq V$ implies $V^{\perp} \subseteq U^{\perp}$*
4. *$\dim V = d$ implies $\dim V^{\perp} = n - d - 1$*

Proof. All of these properties are the direct applications of the Theorem 3.3 to the Proposition 3.2. ■

Now before getting to the examples, we will prove a lemma that is going to allow us to do the calculations easier.

Lemma 3.5. *Let U and V be projective subspaces of $\mathbb{P}^n$ then we have $(U+V)^{\perp} = U^{\perp} \cap V^{\perp}$ and $(U^{\perp} \cap V^{\perp})^{\perp} = (U+V)$.*

Proof. We will first show $(U+V)^{\perp} = U^{\perp} \cap V^{\perp}$. We start off by showing this inclusion $(U+V)^{\perp} \subseteq U^{\perp} \cap V^{\perp}$. We already have $U \subseteq U+V$ and $V \subseteq U+V$ therefore we get $(U+V)^{\perp} \subseteq U^{\perp}$ and $(U+V)^{\perp} \subseteq V^{\perp}$ now if we intersect both of these subset relations we get $(U+V)^{\perp} \subseteq U^{\perp} \cap V^{\perp}$.

Now for the other inclusion take $x \in U^{\perp} \cap V^{\perp}$ then for any $y \in U+V$, we can write $y = [\alpha\hat{u} + \beta\hat{v}]$ for some scalars α, β and affine representatives $\hat{u}, \hat{v}$. Therefore $\langle x, y \rangle_Q = \langle x, \alpha\hat{u} + \beta\hat{v} \rangle_Q = \alpha \langle x, \hat{u} \rangle_Q + \beta \langle x, \hat{v} \rangle_Q = 0$ therefore $x \in (U+V)^{\perp}$.

Now we are going to show $(U^{\perp} \cap V^{\perp})^{\perp} = (U+V)$. Since $L^{\perp\perp} = L$ for any projective subspace L, if we take the polar of both sides of the equation $(U+V)^{\perp} = U^{\perp} \cap V^{\perp}$, we get

$$(U+V) = (U^{\perp} \cap V^{\perp})^{\perp},$$

which is exactly what we wanted to show. ■

Furthermore, we are going to show the observation, we have seen in Example 3.1, which was, when $p \in Q$, we have $p^{\perp} = T_pQ$.

Lemma 3.6. *Let $Q = \mathbb{V}(q)$ be a quadric hypersurface in $\mathbb{P}^n$. Let $p \in Q$ then $p^\perp = T_pQ$.*

Proof. Note that $T_pQ = \mathbb{P}(\ker \nabla q(p))$, and by definition $p^\perp = \{x \in \mathbb{P}^n \mid \sum_{i=0}^n \frac{\partial q}{\partial x_i}(p)x = 0\}$, which is exactly T_pQ. ■

Example 3.3. We are going to consider all possible polars in $\mathbb{P}^3$. Fix the nonsingular quadric Q to be $\mathbb{V}(q)$ given by degree 2 polynomial q. Let $p = (p_0 : p_1 : p_2 : p_3) \in \mathbb{P}^3$ be a point then $p^\perp = \left\{x \in \mathbb{P}^3 \mid \frac{\partial q}{\partial x_0}(p)x_0 + \cdots + \frac{\partial q}{\partial x_3}(p)x_3 = 0\right\}$. We are going to write it as

$$p^\perp = \mathbb{V}\left(\frac{\partial q}{\partial x_0}(p)x_0 + \cdots + \frac{\partial q}{\partial x_3}(p)x_3\right).$$

Let L be a line in $\mathbb{P}^3$ given by the projective span of two points a_1 and a_2. Then $L^\perp = (a_1+a_2)^\perp = a_1^\perp \cap a_2^\perp$ by the lemma 3.5 therefore we can write it as

$$L^\perp = \mathbb{V}\left(\frac{\partial q}{\partial x_0}(p_1)x_0 + \cdots + \frac{\partial q}{\partial x_3}(p_1)x_3\right) \cap \mathbb{V}\left(\frac{\partial q}{\partial x_0}(p_2)x_0 + \cdots + \frac{\partial q}{\partial x_3}(p_2)x_3\right).$$

Finally let H be the hyperplane defined as $\mathbb{V}(b_0x_0 + \cdots + b_3x_3)$, then $H^\perp$ is the point $p \in \mathbb{P}^3$ such that $\left(\frac{\partial q}{\partial x_0}(p), \ldots, \frac{\partial q}{\partial x_3}(p)\right) = (b_0 : \cdots : b_3)$ which is evident by the relation $p^{\perp\perp} = p$.

As an example choose quadratic form $Q = Q_{ED} = \mathbb{V}(q)$ where $q = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Consider the following three points in the projective space

$$a_1 = (1:3:0:0), \quad a_2 = (1:0:3:0), \quad a_3 = (1:0:0:3).$$

The polar of these points are the following respectively

$$\begin{aligned} a_1^\perp &= \mathbb{V}(x_0 + 3x_1) \\ a_2^\perp &= \mathbb{V}(x_0 + 3x_2) \\ a_3^\perp &= \mathbb{V}(x_0 + 3x_3). \end{aligned}$$

Define a flag $\mathcal{K}$ in $\mathbb{P}^3$ by $K_0 \subseteq K_1 \subseteq K_2$, where $K_0 = a_1$, $K_1 = a_1 + a_2$, and $K_2 = a_1 + a_2 + a_3$. We have the dual of the flag as

$$\begin{aligned} K_0^\perp &= a_1^\perp = \mathbb{V}(x_0 + 3x_1) = \mathbb{P}(\text{span}\{(-3:1:0:0), (-3:1:1:0), (-3:1:1:1)\}) \\ K_1^\perp &= a_1^\perp \cap a_2^\perp = \mathbb{V}(x_0 + 3x_1, x_0 + 3x_2) = \mathbb{P}(\text{span}\{(-3:1:1:0), (-3:1:1:1)\}) \\ K_2^\perp &= a_1^\perp \cap a_2^\perp \cap a_3^\perp = \mathbb{V}(x_0 + 3x_1, x_0 + 3x_2, x_0 + 3x_3) = \mathbb{P}(\text{span}\{(-3:1:1:1)\}). \end{aligned}$$

In the Figure 15, the image on the left is the flag K_0, K_1 and K_2 defined as above and the image on the right is the flag $K_2^\perp$, $K_1^\perp$ and $K_0^\perp$.

We note that the Figure 15 does not show the geometric meaning. The reason for that is the

Figure 15: The affine picture of the flag before and after taking the polar.

images in the Figure 15 are in the affine chart, therefore we lose the most of the geometry in $\mathbb{A}^4$. We will see how polarities give notion of orthogonality in affine space in the next subsection. $\triangle$

3.2 Reciprocal Polar Varieties

First, we will define the reciprocal polar varieties and understand the geometry of it by the definition. Afterwards we are going to see an equivalent way to define it.

Definition 3.3. Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety of codimension c, let $i \in \{0, ..., \dim X\}$ and let $K_i \subseteq \mathbb{P}^n$ be a projective subspace of dimension $c + i - 1$. The i-th reciprocal polar variety $W_{K_i}^{\perp}$ is defined to be

$$\{p \in X \setminus K_i^{\perp} \mid T_pX \not\pitchfork (p + K_i^{\perp})^{\perp}\}.$$

Remark 3.3. We will show that varieties $X \subseteq \mathbb{P}^n$ of codimension $c = 1$, the $i = 0$ reciprocal polar variety is the X itself. Since $\dim K_0 = 0$, we get $\dim K_0^{\perp} = n - 1$, therefore $\dim(p + K_0^{\perp}) = n$, hence $\dim((p+K_0^{\perp})^{\perp}) = -1$, finally $(p+K_0^{\perp})^{\perp} = \emptyset$. For each point $p \in X$, we have $(p+K_0^{\perp})^{\perp} = \emptyset$ therefore $W_{K_0}^{\perp}(X) = X$.

Example 3.4. Let the ambient space be $\mathbb{P}^3$, let our projective variety be $X = \mathbb{V}(x_1^2+x_2^2+x_3^2-x_0^2)$, the quadric be $Q_{ED} = \mathbb{V}(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ and the affine chart be $\{x_0 = 1\}$. Let $\mathcal{K}$ to be the flag in Example 3.3 that is for the points

$$a_1 = (1 : 3 : 0 : 0), \quad a_2 = (1 : 0 : 3 : 0), \quad a_3 = (1 : 0 : 0 : 3),$$

the flag $\mathcal{K}$ is $K_0 \subseteq K_1 \subseteq K_2$, where $K_0 = a_1$, $K_1 = a_1 + a_2$, and $K_2 = a_1 + a_2 + a_3$.

Let $p = (p_0 : p_1 : p_2 : p_3) \in X$. We will consider all the reciprocal polar varieties from $i = 0$ through $i = 2$. Several observations specifically for this example are the following. Since X is a hypersurface, T_pX is a hyperplane. Therefore the only way that it has a non-transversal intersection is when $(p + K_i^{\perp})^{\perp} \subseteq T_pX$. By polarity, this is equivalent to $T_pX^{\perp} \in (p + K_i^{\perp})$. Since $\operatorname{codim} X = c = 1$, we have $\dim K_i = i$.

Case $i = 0$**:** Using remark 3.3 we get $W_{K_0}^{\perp}(X) = X$.

Case $i = 1$**:** We have to check the points $p \in X$ where $T_pX^{\perp} \in p + K_1^{\perp}$. Firstly, $T_pX^{\perp} = (-p_0 : p_1 : p_2 : p_3)$ and by the Example 3.3, we get

$$K_1^{\perp} = \mathbb{P}(\text{span}\{(-3 : 1 : 1 : 0), (-3 : 1 : 1 : 1)\})$$

Therefore essentially we are checking the condition that $p \in X$ such that

$$\begin{pmatrix} -p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Assuming the $p \notin K_1$ this relation can be seen as a rank condition of the matrix

$$\begin{pmatrix} -p_0 & p_0 & -3 & -3 \\ p_1 & p_1 & 1 & 1 \\ p_2 & p_2 & 1 & 1 \\ p_3 & p_3 & 0 & 1 \end{pmatrix}.$$

Therefore we are going to check for which points $p = [p_0 : p_1 : p_2 : p_3] \in X$, that matrix is noninvertible. This is the condition that the determinant $2p_0(p_2 - p_1) = 0$. So first reciprocal polar variety is the points $p \in X$ where either $p_0 = 0$ or $p_1 = p_2$. Since we have chosen $x_0 = 0$ to be the hyperplane at infinity in the affine chart $\{x_0 = 1\}$ we get

$$\mathbb{V}(x_1^2 + x_2^2 + x_3^2 - 1, x_1 - x_2).$$

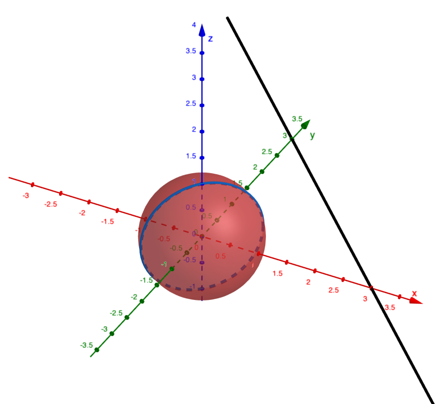

Figure 16: $i = 1$ reciprocal polar variety example

In the Figure 16, the black line passing through $(3,0,0)$ and $(0,3,0)$ is $K_1 = a_1 + a_2$ in the affine chart. The red sphere is the variety X in the affine chart. Finally the blue circle on the sphere is the intersection of $x_1^2+x_2^2+x_3^2-1$ and $x_1 - x_2$, which is the reciprocal polar variety for $i = 1$.

Case $i = 2$**:** See that $K_2^{\perp} = (-3 : 1 : 1 : 1)$ is a point, therefore we are looking at the following membership problem

$$\begin{pmatrix} -p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Similarly to the case $i = 1$, this can be calculated by the 3×3 minors of

$$\begin{pmatrix} -p_0 & p_0 & -3 \\ p_1 & p_1 & 1 \\ p_2 & p_2 & 1 \\ p_3 & p_3 & 1 \end{pmatrix},$$

which are $\{-p_0\,(2p_2 - 2p_3)\,, -p_0\,(2p_1 - 2p_3)\,, -p_0\,(2p_1 - 2p_2)\}$. So we are looking at the points $p \in X$ such that all of the above conditions hold. Observe that in the affine chart $x_0 = 1$, this is exactly the set

$$\mathbb{V}(x_1^2 + x_2^2 + x_3^2 - 1,\ x_2 - x_3,\ x_1 - x_3,\ x_1 - x_2) = \left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \right\}.$$

Therefore, when we plot it in the affine chart we get the following image

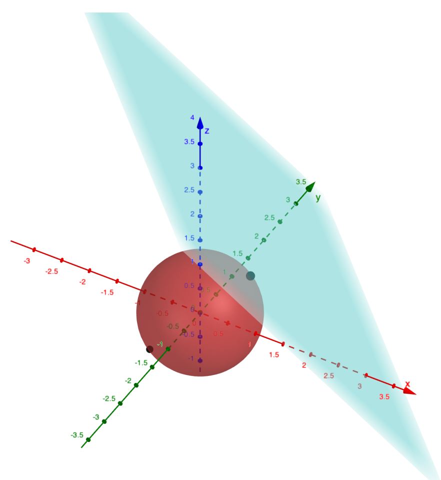

Figure 17: Reciprocal polar variety for K_2

.

In the Figure 17, the blue plane is the $K_2 = a_1 + a_2 + a_3$ and the two black dots on the red sphere is the reciprocal polar variety for $i = 2$. Combining the image of the reciprocal polar variety for K_1 and K_2, we get the following image.

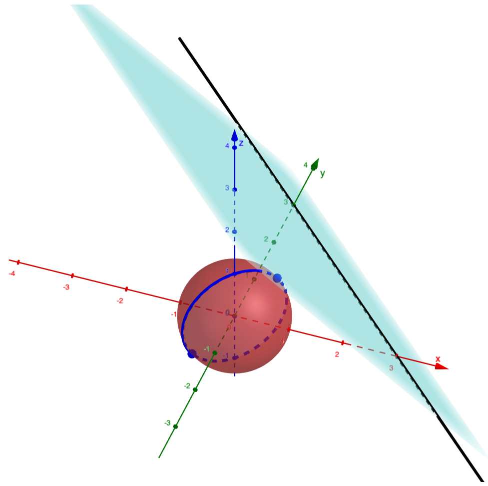

Figure 18: Reciprocal polar varieties with the flag

This image can be interpreted as the following: reciprocal polar variety is the subvariety of X, whose points have the property that normal space at that point and the normal space projective subspace intersect non-transversely. $\triangle$

Now we will work with an example where the flag $\mathcal{K}$ itself lives inside the hyperplane at infinity H_∞. The first will be a degenerate example, and the second will be a non-degenerate example.

Example 3.5. Let $X \subseteq \mathbb{P}^3$ be a projective variety, hyperplane at infinity H_∞ and the flag $\mathcal{K}$ as before. Now to impose the condition of $K_i \subseteq H_\infty$ for each $i = 0, 1, 2$ we intersect K_i with the H_∞. That is the flag we have now is the following:

$$\mathbb{P}(\mathrm{span}\{(0,3,0,0)\}) \subseteq \mathbb{P}(\mathrm{span}\{(0,3,0,0),(0,0,3,0)\}) \subseteq \mathbb{P}(\mathrm{span}\{(0,3,0,0),(0,0,3,0),(0,0,0,3)\}) \subseteq \mathbb{P}^3.$$

As in the previous example, we will work case by case.

Case $i = 0$: We have $W^\perp_{K_0}(X) = X$ by the remark that was established.

Case $i = 1$: See that $K_1^\perp = \{y \in \mathbb{P}^3 : \forall \lambda_1, \lambda_2 \in \mathbf{k} :\ 3\lambda_1 y_1 + 3\lambda_2 y_2 = 0\} = \mathbb{P}(\mathrm{span}\{(1,0,0,0),(0,0,0,1)\})$. Now the condition $T_pX^\perp \in p + K_1^\perp$ translates to

$$\begin{pmatrix} -p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \in \mathrm{span}\left\{ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

But for any $p \in X$ this condition holds trivially. Therefore $W^\perp_{K_1}(X) = X$.

Case $i = 2$**:** Similarly to the case $i = 1$ we get $K_2^\perp = \{y \in \mathbb{P}^3 : \forall \lambda_1, \lambda_2, \lambda_3 \in \mathbf{k} : \; 3\lambda_1 y_1 + 3\lambda_2 y_2 + 3\lambda_3 y_3 = 0\} = \mathbb{P}(\mathrm{span}\{(1,0,0,0)\})$. Now the condition $T_pX^\perp \in p + K_2^\perp$ translates to

$$\begin{pmatrix} -p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \in \mathrm{span}\left\{ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

which again trivially holds for any $p \in X$. Hence $W_{K_2}^\perp(X) = X$ as well.

So for each of the cases we got the same variety for the reciprocal as well. The reason for this behavior will become evident in the theorem at the end of this section. △

Before moving on, we are going to do the same calculation where we change the variety X to be the ellipsoid and see the geometry that is happening.

Example 3.6. Our ambient space is $\mathbb{P}^3$, projective variety is $X = \mathbb{V}(-x_0^2 + c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2)$ where c_1, c_2, c_3 are positive real numbers. Let the quadric $Q = Q_{ED} = \mathbb{V}(x_0^2 + x_1^2 + x_2^2 + x_3^2)$. Choose the flag $\mathcal{K}$ to be the same as before

$$\mathbb{P}(\mathrm{span}\{(0,3,0,0)\}) \subseteq \mathbb{P}(\mathrm{span}\{(0,3,0,0),(0,0,3,0)\}) \subseteq \mathbb{P}(\mathrm{span}\{(0,3,0,0),(0,0,3,0),(0,0,0,3)\}) \subseteq \mathbb{P}^3.$$

For $p = (p_0 : p_1 : p_2 : p_3) \in X$ we see that the tangent space at the point p is

$$T_pX = \mathbb{V}(-p_0 x_0 + c_1 p_1 x_1 + c_2 p_2 x_2 + c_3 p_3 x_3)$$

and its polar is

$$T_pX^\perp = (-p_0 : c_1 p_1 : c_2 p_2 : c_3 p_3).$$

We will do case-by-case analysis for the calculation of the reciprocal polar variety. As before by the remark 3.3, $W_{K_0}^\perp(X) = X$ therefore interesting picture happens at $i = 1$ and $i = 2$.

Case $i = 1$**:** By our previous calculation $K_1^\perp = \mathbb{P}(\mathrm{span}\{(1,0,0,0),(0,0,0,1)\})$ therefore the condition $T_pX^\perp \in p + K_1^\perp$ is

$$\begin{pmatrix} -p_0 \\ c_1 p_1 \\ c_2 p_2 \\ c_3 p_3 \end{pmatrix} \in \mathrm{span}\left\{ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This condition is satisfied for those $p \in X$ such that the determinant of the augmented matrix

$$\begin{pmatrix} -p_0 & p_0 & 1 & 0 \\ c_1 p_1 & p_1 & 0 & 0 \\ c_2 p_2 & p_2 & 0 & 0 \\ c_3 p_3 & p_3 & 0 & 1 \end{pmatrix}$$

is 0. When the determinant is calculated we see that this is the condition of $p \in X$ such that $p_1 p_2 (c_1 - c_2) = 0$. Therefore for $i = 1$ the reciprocal polar variety is

$$W_{K_1}^{\perp}(X) = \mathbb{V}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 - x_0^2, x_1 x_2).$$

In the affine chart $\{x_0 = 1\}$, this is the curve:

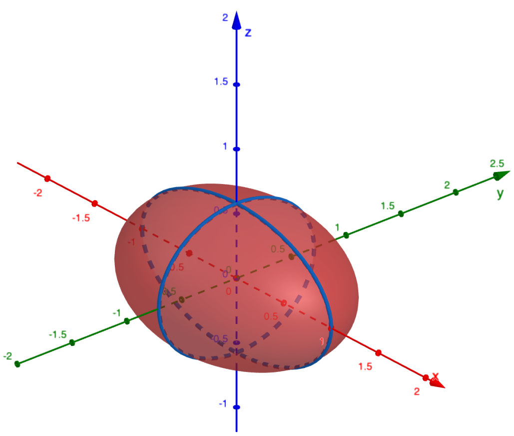

Figure 19: First reciprocal polar variety for $c_1 = 1$, $c_2 = 2$, $c_3 = 3$

Case $i = 2$ Similarly $K_2^{\perp} = \mathbb{P}(\text{span}\{(1,0,0,0)\})$ therefore the condition $T_p X^{\perp} \in p + K_2^{\perp}$ is

$$\begin{pmatrix} -p_0 \\ c_1 p_1 \\ c_2 p_2 \\ c_3 p_3 \end{pmatrix} \in \text{span} \left\{ \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

So we are looking for the points $p \in X$ such that 3×3 minors of augmented matrix

$$\begin{pmatrix} -p_0 & p_0 & 1 \\ c_1 p_1 & p_1 & 0 \\ c_2 p_2 & p_2 & 0 \\ c_3 p_3 & p_3 & 0 \end{pmatrix}$$

vanishes. The 3×3 minors of that matrix are $\{p_2 p_3(c_2 - c_3), p_1 p_3(c_1 - c_3), p_1 p_2(c_1 - c_2)\}$. Therefore the second reciprocal polar variety is

$$W^{\perp}_{K_2}(X) = \mathbb{V}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 - x_0^2, x_2 x_3(c_2 - c_3), x_1 x_3(c_1 - c_3), x_1 x_2(c_1 - c_2)).$$

A case-by-case analysis shows that, in the affine chart $x_0 = 1$, the second reciprocal polar variety consists of six points given by the intersection of ellipsoid with the coordinate axes.

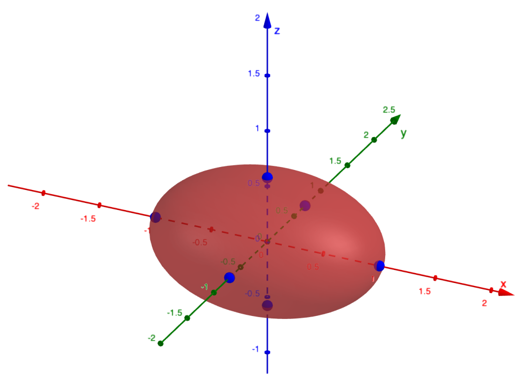

Figure 20: Second reciprocal polar variety for $c_1 = 1$, $c_2 = 2$, $c_3 = 3$

Now if we combine both of the pictures, we see the inclusions of first and second polar varieties as well.

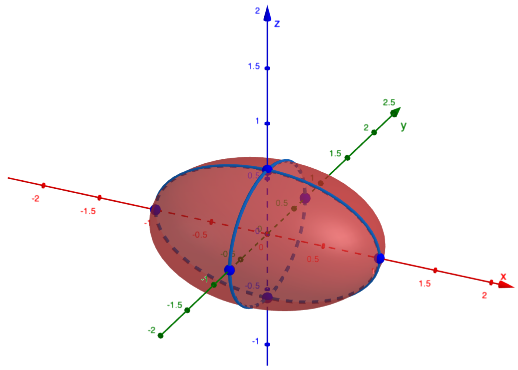

Figure 21: First and second reciprocal polar varieties for $c_1 = 1$, $c_2 = 2$, $c_3 = 3$

Note that the images do not include the flag, since the flag lies in the hyperplane at infinity. Its role in these pictures is to describe directions. △

Lemma 3.7. *Let $p \in T_pX^\perp$ then $p \in Q$.*

Proof. Let $p \in T_pX^\perp$ then by definition $\langle p, v\rangle_Q = 0$ for every $v \in T_pX$. By Lemma 2.5 we know that $p \in T_pX$ therefore, in particular we get $\langle p, p\rangle_Q = 0$. Expanding this we get $\sum_{i=0}^{n} \frac{\partial q}{\partial x_i}(p)p_i = 0$. Since q is degree two homogeneous polynomial by Euler's relation 2.4 we get $2q(p) = 0$, which implies $p \in Q$. ∎

Lemma 3.8. *If X intersects Q transversely, then for all points $p \in X$, we have $p \notin T_pX^\perp$.*

Proof. Let $p \in X$ then we will consider two cases. If $p \notin Q$ then by Lemma 3.7 we have $p \notin T_pX^\perp$. If $p \in Q$, then $p \in X \cap Q$. By transversality assumption on X and Q, we have

$$T_pX + T_pQ = \mathbb{P}^n. \tag{13}$$

By Lemma 3.6 for $p \in Q$, we have $T_pQ = p^\perp$. After taking polar of both sides of the equation (13), we get

$$\begin{aligned}\emptyset = (\mathbb{P}^n)^\perp &= (T_pX + T_pQ)^\perp \\ &= (T_pX + p^\perp)^\perp \\ &= (T_pX^\perp \cap p)^{\perp\perp} \\ &= T_pX^\perp \cap p,\end{aligned}$$

which implies $p \notin T_pX^\perp$. ∎

Remark 3.4. We are going to show an equivalent way of defining the reciprocal polar varieties, where Q intersects X transversely which implies $p \notin T_pX^\perp$ by Lemma 3.8. Let $p \notin K_i^\perp$, we start off by finding the dimension of $(p + K_i^\perp)^\perp$. By assumption $\dim K_i = i + n - m - 1$ therefore $\dim K_i^\perp = m - i$, hence $\dim(K_i^\perp + p) = m - i + 1$, finally $\dim((K_i^\perp + p)^\perp) = n - m + i - 2$.

The condition $T_pX \not\pitchfork (p + K_i^\perp)^\perp$ which means $\dim(T_pX + (p + K_i^\perp)^\perp) < n$ is equivalent by the Grassmann formula 2.3 to

$$\begin{aligned} n &> \dim T_pX + \dim(p + K_i^\perp)^\perp - \dim(T_pX \cap (p + K_i^\perp)^\perp) \\ &= m + (n - m + i - 2) - \dim(T_pX \cap (p + K_i^\perp)^\perp). \end{aligned}$$

By reordering we get the other equivalence

$$\dim(T_pX \cap (p + K_i^\perp)^\perp) > i - 2$$

which is

$$\dim(T_pX \cap (p + K_i^\perp)^\perp) \geq i - 1$$

Now using Lemma 3.5, we have following equalities

$$T_pX \cap (p + K_i^\perp)^\perp = T_pX \cap p^\perp \cap K_i = (T_pX^\perp + p)^\perp \cap K_i = (T_pX^\perp + p + K_i^\perp)^\perp$$

therefore

$$\dim((T_pX^\perp + p + K_i^\perp)^\perp) \geq i - 1$$

which is equivalent to

$$\dim((T_pX^\perp + p) + K_i^\perp) \leq n - i.$$

We are going to calculate of the dimension $\dim(T_pX^\perp + p)$. Since $\dim(T_pX) = m$ by taking polar we get $\dim(T_pX^\perp) = n - m - 1$. As $p \notin T_pX^\perp$ we get $\dim(T_pX^\perp + p) = n - m$. Now using the Grassmann formula again, we get:

$$\begin{aligned} n - i &\geq \dim((T_pX^\perp + p)) + \dim(K_i^\perp) - \dim((T_pX^\perp + p) \cap K_i^\perp) \\ &= (n - m) + (m - i) - \dim((T_pX^\perp + p) \cap K_i^\perp) \\ &= n - i - \dim((T_pX^\perp + p) \cap K_i^\perp) \end{aligned}$$

which is therefore equivalent to $\dim\left((T_pX^\perp + p) \cap K_i^\perp\right) \geq 0$ or equivalently, $(T_pX^\perp + p) \cap K_i^\perp \neq \emptyset$.

Therefore, we can write the i-th reciprocal polar variety as either

$$W_{K_i}^\perp(X) = \left\{p \in X \setminus K_i^\perp \mid \dim\left((T_pX^\perp + p) \cap K_i^\perp\right) \geq 0\right\}$$

or

$$W^{\perp}_{K_i}(X) = \left\{p \in X \setminus K_i^{\perp} \mid (T_pX^{\perp} + p) \cap K_i^{\perp} \neq \emptyset\right\}.$$

So in the case of $i = m$, K_m is a hyperplane and $K_m^{\perp}$ is a point. Assuming that it is not a point in the variety X it is

$$W^{\perp}_{K_m}(X) = \left\{p \in X \mid K_m^{\perp} \in (T_pX^{\perp} + p)\right\}.$$

3.3 Affine Reciprocal Polar Varieties

In the previous subsection, we have shown some examples of the affine reciprocal polar varieties. In this subsection we are going to make it precise and show why it is also useful to think about them.

First we note that the image for the affine reciprocal polar varieties changes when the flag is contained in the hyperplane at infinity. Our first goal is to explain this phenomenon. In order to explain it we first give an example of how intersection with hyperplane at infinity takes records the directions.

Example 3.7. Let the ambient space be $\mathbb{P}^2$, the quadric $Q = Q_{ED} = \mathbb{V}(x_0^2 + x_1^2 + x_2^2)$ and the hyperplane at infinity $H_\infty = \mathbb{V}(x_0)$. Recall that we can decompose projective space as $\mathbb{P}^2 = \mathbb{A}^2 \cup H_\infty$, where H_∞ keeps track of the directions. So given $X \subseteq \mathbb{P}^2$, when we intersect it with hyperplane at infinity H_∞ which is denoted as X_∞, it keeps track of the possible directions that $Y = X \cap \mathbb{A}^2$ reach infinity. For example, for $X = \mathbb{V}(x_1x_2 - x_0^2)$, we have $Y = \mathbb{V}(x_1x_2 - 1)$. and $X_\infty = \mathbb{V}(x_1x_2)$. If we plot these, we see that in the $\mathbb{A}^2$ there is the Y and the 2 points at the infinity.

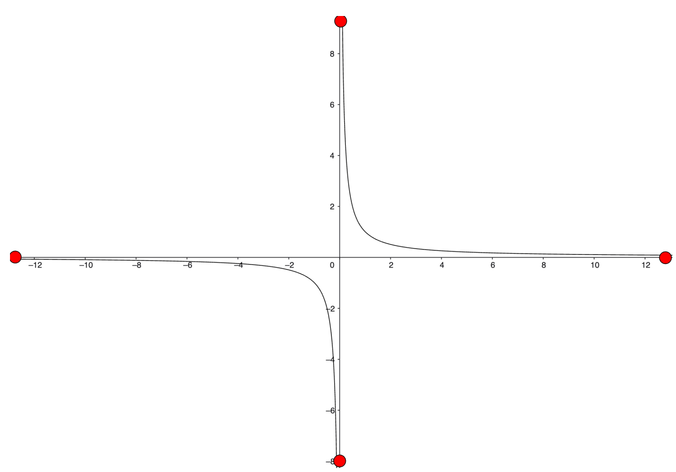

Figure 22: $xy = 1$ and the two points at infinity in red.

The reason we are saying two points at infinity although there are four red dots in the image is because the points at H_∞ are the lines. So two dots in the image defines a one line that corresponds to the line that goes to infinity in that direction. $\triangle$

Now using the idea of having directions in H_∞ we are going to define the normal space at a point.

Definition 3.4. Let Q be quadric, H_∞ be a hyperplane at infinity, and L be a projective subspace in $\mathbb{P}^n$, not contained in H_∞. Take $p \in L$ then the normal space of L at p denoted as $L_p^\perp \subseteq \mathbb{P}^n$ is defined as

$$L_p^\perp = p + (L \cap H_\infty)^\perp.$$

Where polarity $\perp$ on $(L \cap H_\infty)$ is the induced polarity by $Q_\infty := Q \cap H_\infty$. Since $H_\infty \simeq \mathbb{P}^{n-1}$ and $(L \cap H_\infty) \subseteq \mathbb{P}^{n-1}$, we can think of Q_∞ as the quadric defined for $\mathbb{P}^{n-1}$ which are the directions of $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$.

Example 3.8. In this example we are going to use the observation that when the $Q = Q_{ED}$ the polarity defines an orthogonal complement in the projective space. Suppose we are in $\mathbb{P}^3$ and $Q = Q_{ED} = \mathbb{V}(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ and $L = \mathbb{V}(c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3)$. When intersected with the hyperplane at infinity $Q_\infty = \mathbb{V}(x_1^2 + x_2^2 + x_3^2)$ and $L \cap H_\infty = \mathbb{V}(c_1x_1 + c_2x_2 + c_3x_3)$. Therefore the polar is

$$(L \cap H_\infty)^\perp = (0 : c_1 : c_2 : c_3)$$

since this is a hyperplane in $H_\infty \simeq \mathbb{P}^2$. Finally when we projective span with the point p we get

$$L_p^\perp = \mathbb{P}(\text{span}\{(p_0, p_1, p_2, p_3), (0, c_1, c_2, c_3)\}).$$

Notice that we lost the c_0 in the expression of $L_p^\perp$ however it makes sense since c_0 only accounts for the translation of L in the affine chart $\mathbb{A}^3 = \mathbb{P}^3 \setminus H_\infty$ and it does not change the normal space.

Say $c_0 = -3$ and $c_1 = c_2 = c_3 = 1$ then clearly $p = (1 : 1 : 1 : 1) \in L$. The image below is the $L \cap \mathbb{A}^3$ and $L_p^\perp \cap \mathbb{A}^3$ together.

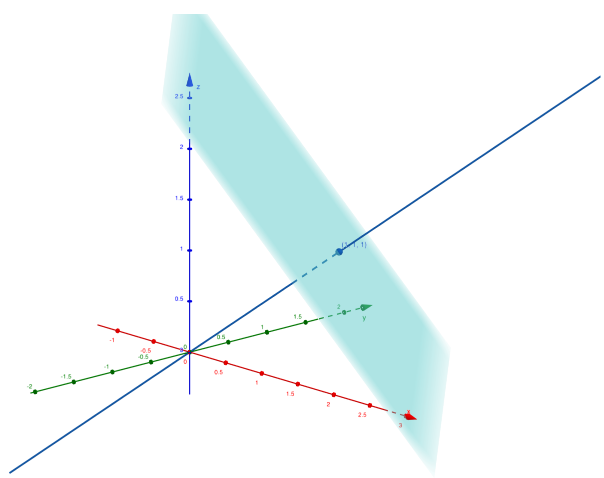

Figure 23: $L \cap \mathbb{A}^3$ and $L_p^\perp \cap \mathbb{A}^3$

The reason it gives the correct normal space at that point p can be seen as follows. Intersecting

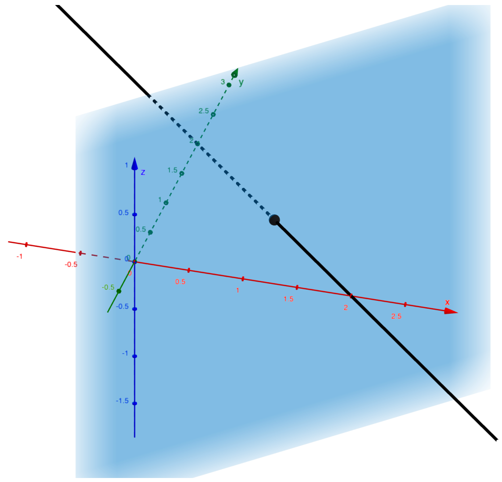

Figure 24: Normal space of a line passing through A and B at the point p

L with H_∞ records all possible directions in which L goes to infinity, viewed in $L \cap H_\infty \subseteq \mathbb{P}^2$. We have already observed that when $Q = Q_{ED}$ the polarity coincides with the orthogonal complement in affine cone. Therefore, $(L \cap H_\infty)^\perp$ is the orthogonal complement of the L in the space of directions. Finally, taking projective sum with the point $p \in L$ just allows us to move the $(L \cap H_\infty)^\perp$ to the $p \in L$ in the affine chart $\mathbb{A}^3$. △

Example 3.9. Consider another example where L is now a line in $\mathbb{P}^3$. Let $A = (1 : a_1 : a_2 : a_3)$ and $B = (1 : b_1 : b_2 : b_3)$. A line L spanned by A and B intersects the hyperplane at infinity at $L \cap H_\infty = (0 : a_1 - b_1 : a_2 - b_2 : a_3 - b_3)$. Now taking polar with $Q = Q_{ED}$ gives the hyperplane

$$\mathbb{V}((a_1 - b_1)x_1 + (a_2 - b_2)x_2 + (a_3 - b_3)x_3). \tag{14}$$

The normal space of L at the point p is therefore (14) translated to $p = (1 : p_1 : p_2 : p_3)$ which is

$$\mathbb{V}((a_1 - b_1)(x_1 - p_1x_0) + (a_2 - b_2)(x_2 - p_2x_0) + (a_3 - b_3)(x_3 - p_3x_0)).$$

As an example choose $A = (1 : 2 : 0 : 0)$ and $B = (1 : 0 : 2 : 0)$ then $p = (1 : 1 : 1 : 0) \in L$ as well. So if we plug in the variables for the normal space we get the equation $\mathbb{V}(x_2 - x_1)$ which can be seen from the Figure 24 as well. △

With this construction we are going to define the Euclidean normal space of a projective variety at a point.

Definition 3.5. Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety, and $p \in X$ be a point. Let quadric $Q = Q_{ED} = \mathbb{V}(x_1^2 + \cdots + x_n^2 + x_0^2)$. Let $H_\infty \subseteq \mathbb{P}^n$ be a hyperplane at infinity. We say $N_pX = p + (T_pX \cap H_\infty)^\perp$ is the Euclidean normal space at the point p.

Remark 3.5. In the affine chart $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$, the Euclidean normal space at the point p N_pX really does give the normal space. So we have a notion of normal space in the projective setting

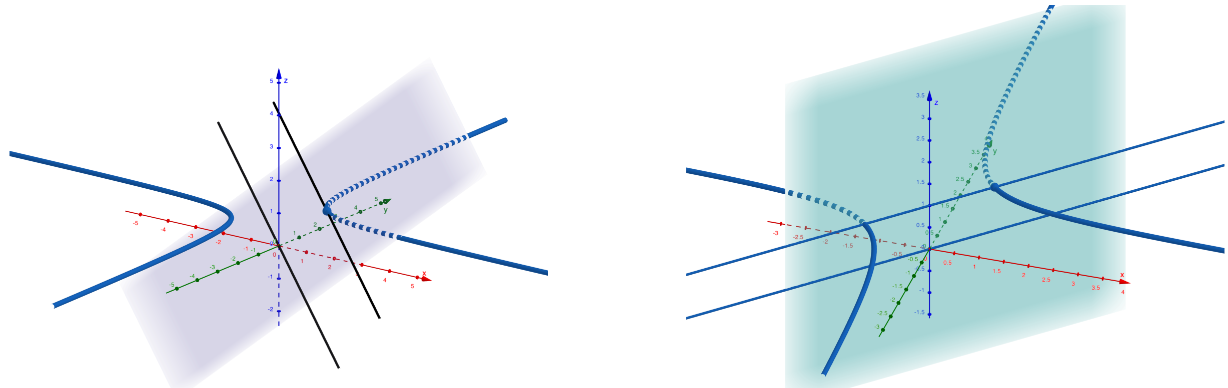

Figure 25: Tangent space and the normal space in the affine cone

using this definition. The reason we are getting the normal space is the following. First T_pX is the tangent space the point p and when intersected with H_∞ it records the directions in $H_\infty \simeq \mathbb{P}^{n-1}$ where T_pX is reaching. Since polarity in the quadric $Q = Q_{ED}$ gives a notion of orthogonal complement $(T_pX \cap H_\infty)^\perp$ is the direction of the orthogonal complement of the tangent space. Therefore in order to move the normal space to point p we projective sum with the p as well.

Example 3.10. Let $X \subseteq \mathbb{P}^2$ be the projective curve defined by $f = x_1x_2 - x_0^2 = 0$. Consider the point $p = (1:1:1) \in X$, we are going to compute the Euclidean normal space of X at the point p N_pX.

First we will find the tangent space T_pX which is given by the projectivization of the kernel of $\nabla f(p) = (-2, 1, 1)$. Therefore $T_pX = \mathbb{V}(-2x_0 + x_1 + x_2)$, in the affine chart $x_0 = 1$ this is the line $x_1 + x_2 = 2$, that is a line through $(2, 0)$ and $(0, 2)$. We have already computed the normal space at $(1, 1)$ of the line passing through $(2, 0)$ and $(0, 2)$ in Example 3.9. The only difference now is previous example was embedded in $\mathbb{P}^3$ using $x_3 = 0$. Therefore the normal space is $N_pX = \mathbb{V}(2(x_1 - x_0) - 2(x_2 - x_0)) = \mathbb{V}(2x_1 - 2x_2) = \mathbb{V}(x_1 - x_2)$. The Figure 25 shows the tangent space and the normal space in the affine cone $\mathbb{A}^3$ for this example. $\triangle$

In the Figure 25, the blue curve is the $xy = 1$ in the affine cone $\mathbb{A}^3$ of $\mathbb{P}^2$ on $z = 1$. The black line is the affine tangent space and the blue plane is the projective tangent space. Therefore after intersecting with hyperplane at infinity, which is $z = 0$ in that picture we get the image on the left. Now since when $Q = Q_{ED}$, taking polars is the notion of orthogonal complement in the affine chart, we get the normal space in $z = 0$ in the image on the right. After we take the projective span with the point in the variety we get the turquoise of the normal space. Finally, when intersected with $z = 1$ we get the exactly the normal space in the affine picture.

Remark 3.6. In our previous definition of reciprocal polar varieties there was no assumption on the flag $\mathcal{K}$. That is why when we were looking at the affine chart the images were not directly corresponding to the normal spaces. Now we are going to define the affine reciprocal polar varieties with the assumption of every element in the flag $\mathcal{K}$ is contained in the hyperplane at infinity H_∞.

Therefore the condition with $p + K_i^\perp$ will be actually the normal space of K_i since we have $K_i \cap H_\infty = K_i$.

Definition 3.6. Let $H_\infty \subset \mathbb{P}^n$ be the hyperplane at infinity and let $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$. We consider the affine part of projective variety X as $Y = X \cap \mathbb{A}^n$. Assume that every projective subspace K in the flag $\mathcal{K}$ is contained in the H_∞. Take $i = 0, \dots, m$, let $L = K_i \subseteq H_\infty$ be a linear space of dimension $c+i-1$. We define affine reciprocal polar variety to be the affine part of the reciprocal polar variety, that is

$$W_L^\perp(Y) := W_L^\perp(X) \cap \mathbb{A}^n$$

where $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$.

Remark 3.7. We will assume that $H_\infty = \mathbb{V}(x_0)$ for the ease of computation and visualization. However one could have taken last element K_{n-1} of the flag $\mathcal{K}$ to be the H_∞ itself, then there would not be any confusion. However in our previous examples we sometimes tried to visualize the reciprocal polar variety in an affine chart where $\mathcal{K}$ was not in the hyperplane at infinity.

Remark 3.8. In the setting of reciprocal polar varieties note that projective subspace $(p+L^\perp)^\perp$ is contained in the hyperplane H_∞ since $L \subseteq H_\infty$ therefore $H_\infty^\perp \subseteq L^\perp$ hence $H_\infty^\perp \subseteq p + H_\infty^\perp \subseteq p + L^\perp$ finally

$$(p + L^\perp)^\perp \subseteq H_\infty.$$

Therefore we can define the affine cone $I_{p,L^\perp}$ of $(p + L^\perp)^\perp$ to be a linear subspace in the affine cone $\mathbb{A}^n$ of $H_\infty \subseteq \mathbb{P}^{n-1}$.

Proposition 3.9. *[5] Affine reciprocal polar variety can be written as*

$$W_L^\perp(Y) = \{p \in Y \setminus L^\perp \mid T_pY \not\pitchfork I_{p,L^\perp}\}$$

where T_pY is the affine tangent space at p, centered at the origin.

We refer [5, 2.2. Generalized polar varieties] for the details of why these two affine reciprocal polar varieties are the same. However we will demonstrate the theorem in an example and show what we mean by the affine cones of $(p + L^\perp)^\perp$.

Example 3.11. Consider the continuation of the Example 3.6. Note that the flag $\mathcal{K}$ is contained in the hyperplane at infinity $H_\infty = \mathbb{V}(x_0)$. We have calculated the reciprocal polar varieties through some rank conditions on a matrix. In this example we are going to give an explicit description of $(p + K_i^\perp)^\perp$ for $i = 1, 2$ as the projectivization of a linear subspace using p.

As a reminder we have $K_0 \subset K_1 \subset K_2 = H_\infty \subset \mathbb{P}^3$ and by taking polars and projective sums we get

$$(p + K_0^\perp)^\perp \subset (p + K_1^\perp)^\perp \subset (p + K_2^\perp)^\perp \subset H_\infty.$$

By dimension considerations we have already shown that $(p + K_0^{\perp})^{\perp} = \emptyset$. Therefore we will work with $i = 1$ and $i = 2$. For $i = 1$ we have already written

$$p + K_1^{\perp} = \mathbb{P}\left(\operatorname{span}\left\{\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right).$$

Therefore taking another dual of it we must have

$$(p + K_1^{\perp})^{\perp} = \{y \in \mathbb{P}^3 : \forall \lambda_1, \lambda_2, \lambda_3 \in \mathbf{k} : \ (\lambda_1 p_0 + \lambda_2) y_0 + \lambda_1 p_1 y_1 + \lambda_1 p_2 y_2 + (\lambda_1 p_3 + \lambda_3) y_3 = 0\}.$$

If we rearrange it with $\lambda_1, \lambda_2, \lambda_3$ we must have three equations

$$\begin{aligned} \lambda_1(p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3) &= 0 \\ \lambda_2(y_0) &= 0 \\ \lambda_3(y_3) &= 0. \end{aligned}$$

Hence the elements $y \in (p + K_1^{\perp})^{\perp}$ must satisfy $y_0 = 0, y_3 = 0, y_1 = -p_2, y_2 = p_1$ which implies

$$(p + K_1^{\perp})^{\perp} = \mathbb{P}\left(\operatorname{span}\left\{\begin{pmatrix} 0 \\ -p_2 \\ p_1 \\ 0 \end{pmatrix}\right\}\right).$$

Similarly for $i = 2$ we have

$$p + K_2^{\perp} = \mathbb{P}\left(\operatorname{span}\left\{\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}\right).$$

So as a set we have

$$(p + K_2^{\perp})^{\perp} = \{y \in \mathbb{P}^3 \mid \forall \lambda_1, \lambda_2 \in \mathbf{k} : \ (\lambda_1 p_0 + \lambda_2) y_0 + \lambda_1 p_1 y_1 + \lambda_1 p_2 y_2 + \lambda_1 p_3 y_3 = 0\}.$$

After rearranging in terms of λ_1 and λ_2 we get

$$\lambda_1(p_0 y_0 + p_1 y_1 + p_2 y_2 + p_3 y_3) = 0 \lambda_2(y_0) = 0$$

therefore

$$(p+K_2^\perp)^\perp = \mathbb{P}\left(\operatorname{span}\left\{\begin{pmatrix}0\\-p_2\\p_1\\0\end{pmatrix},\begin{pmatrix}0\\-p_3\\0\\p_1\end{pmatrix}\right\}\right)$$

Now coming back to the theorem the affine cones are $I_{p,K_1^\perp} = \operatorname{span}(-p_2,p_1,0) \subseteq \mathbb{A}^3$ and $I_{p,K_2^\perp} = \operatorname{span}\{(-p_2,p_1,0),(-p_3,0,p_1)\}$. Our initial X was the projective ellipsoid therefore $Y = \mathbb{V}(c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - 1)$. The affine tangent plane of Y at the point p denoted as T_pY is given by

$$T_pY = \ker(c_1p_1, c_2p_2, c_3p_3) = \mathbb{V}(c_1p_1x_1 + c_2p_2x_2 + c_3p_3x_3)$$

Now for $i=1$, $I_{p,K_1^\perp}$ intersects T_pY non-transversely if and only if $I_{p,K_1^\perp} \subseteq T_pY$ which is equivalent to basis vector of $I_{p,K_1^\perp}$ evaluates to 0 under the linear functional $c_1p_1x_1+c_2p_2x_2+c_3p_3x_3$ defining T_pY. Therefore we must have

$$c_1p_1(-p_2) + c_2p_2p_1 = 0$$

This is the condition $p_1p_2(c_2-c_1) = 0$. Since $c_2 \neq c_1$ we must have either p_1 or p_2 equal 0. Therefore

$$W^\perp_{K_1}(Y) = Y \cap \mathbb{V}(x_1x_2)$$

which is the exact affine reciprocal polar variety we have found in the example 3.6 with the image 19.

Now for the case $i=2$ since both T_pY and $I_{p,K_2^\perp}$ are 2 dimensional subspaces of $\mathbb{A}^3$ the only way they can intersect non-transversely is when they are exactly the same. Two planes are the same if their normal vectors are proportional.

The normal vector of affine tangent space T_pY is (c_1p_1, c_2p_2, c_3p_3). The normal vector of $I_{p,K_2^\perp}$ is cross product of $(-p_2,p_1,0)$ and $(-p_3,0,p_1)$, which is (p_1^2, p_1p_2, p_1p_3). We are interested in the direction therefore can divide it by p_1 to get $(p_1 : p_2 : p_3)$ for the normal direction of $I_{p,K_2^\perp}$.

Now these two normal vectors are proportional to each other if and only if rank of 2×3 matrix is 1:

$$\operatorname{rk}\begin{pmatrix}c_1p_1 & c_2p_2 & c_3p_3\\ p_1 & p_2 & p_3\end{pmatrix} = 1.$$

When we calculate the 2×2 minors of this matrix we get the following system:

$$\begin{aligned} p_2p_3(c_2-c_3) &= 0\\ p_1p_3(c_1-c_3) &= 0\\ p_1p_2(c_1-c_2) &= 0. \end{aligned}$$

Therefore the affine reciprocal polar variety is $W_{K_2^\perp}(Y) = Y \cap (\mathbb{V}(x_2x_3) \cup \mathbb{V}(x_1x_3) \cup \mathbb{V}(x_1x_2))$.

Which is the exactly what we got in the Example 3.6 and the Figure 20. This calculation also explains why $c_1 = 1, c_2 = 1, c_3 = 1$ in example 3.5 was a degenerate example. $\triangle$

We now arrive at the theorem that allows us to connect top reciprocal polar variety to the Euclidean distance optimization problem.

Proposition 3.10. *Let $X = \mathbb{V}(F_1, \dots, F_s) \subset \mathbb{P}^n$ be a variety of codimension c (where $s \geq c$), and let $Q = \mathbb{V}(q)$ be a quadric. Let $Y = X \cap \mathbb{A}^n$ be its affine part $Y = \mathbb{V}(f_1, \dots, f_s)$, where $\mathbb{A}^n = \mathbb{P}^n \backslash H_\infty$ and $H_\infty = \mathbb{V}(x_0)$. Then the affine reciprocal polar variety $W^{\perp}_{H_\infty}(Y)$ is the closure of the set of smooth points of Y where the $(c+1)$-minors of the augmented Jacobian matrix vanish. That is*

$$\langle f_1, \dots, f_s \rangle + \left\langle (c+1) \times (c+1) \textit{ minors of } \begin{pmatrix} \frac{\partial q}{\partial x_1} & \cdots & \frac{\partial q}{\partial x_n} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_s}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_n}. \end{pmatrix} \right\rangle$$

Proof. Let $\mathcal{J}_Y(p)$ be the $s \times n$ Jacobian matrix of Y evaluated at the point p. Since X is smooth the rank of the Jacobian matrix at p is c. The translated affine tangent space is the kernel of the Jacobian matrix, that is $T_pY = \ker \mathcal{J}_Y(p)$. By Lemma 3.5 we have $(p + H_\infty^\perp)^\perp = p^\perp \cap H_\infty$. Since p is a point

$$p^\perp = \mathbb{V}(\sum_{i=0}^{n} \frac{\partial q}{\partial x_i}(p)x_i)$$

and since $H_\infty = \mathbb{V}(x_0)$ we get that

$$p^\perp \cap H_\infty = \mathbb{V}(\sum_{i=1}^{n} \frac{\partial q}{\partial x_i}(p)x_i) \subseteq H_\infty \simeq \mathbb{P}^{n-1}$$

Therefore the affine cone $I_{p,H_\infty^\perp} \subseteq \mathbb{A}^n$ is defined as

$$I_{p,H_\infty^\perp} = \mathbb{V}(\sum_{i=1}^{n} \frac{\partial q}{\partial x_i}(p)x_i) \subseteq \mathbb{A}^n.$$

Since Q is nonsingular, the gradient vector $\nabla q(p) \neq 0$, meaning $I_{p,H_\infty^\perp}$ is a hyperplane of dimension $n-1$ in affine cone $\mathbb{A}^n$ of $H_\infty \simeq \mathbb{P}^{n-1}$

Since $I_{p,H_\infty^\perp}$ is a hyperplane, the only way $T_pY \not\pitchfork I_{p,H_\infty^\perp}$ is to have $T_pY \subseteq I_{p,H_\infty^\perp}$, which is equivalent to saying $\ker(\mathcal{J}_Y(p)) \subseteq \ker(\nabla q(p))$. Now by linear algebra this holds if and only if

$\nabla q(p)$ is in the row span of the Jacobian matrix $\mathcal{J}_Y(p)$. If we write the augmented matrix

$$M = \begin{pmatrix} \frac{\partial q}{\partial x_1}(p) & \cdots & \frac{\partial q}{\partial x_n}(p) \\ \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_s}{\partial x_1}(p) & \cdots & \frac{\partial f_s}{\partial x_n}(p) \end{pmatrix}$$

we have the following observation: Since $\operatorname{rk}(\mathcal{J}_Y(p)) = c$, if $\nabla q(p) \in \operatorname{row} \mathcal{J}_Y(p)$ then $\operatorname{rk} M = c$ otherwise $\operatorname{rk} M = c+1$. So the non-transversal intersection holds if and only if $\operatorname{rk} M \leq c$, which is equivalent to vanishing of all $(c+1) \times (c+1)$ minors of M. ■

Example 3.12. [28, Example 4.1] Assume q is given by $q(x_0, x_1, \dots, x_n) = x_0^2 + \sum_{i=1}^n (x_i - u_i x_0)^2$ for some sufficiently generic $u_1, \dots, u_n$. Then q restricts to the $\sum_{i=1}^n (x_i - u_i)^2 + 1$ in $\mathbb{A}^n = \mathbb{P}^n \backslash H_\infty$. Note that it differs from the squared Euclidean distance by constant 1, hence has the same critical points. The affine reciprocal polar variety is given by the vanishing of the $(c+1) \times (c+1)$ minors of the matrix

$$\begin{pmatrix} x_1 - u_1 & \cdots & x_n - u_n \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n}. \end{pmatrix}$$

and the points in the variety. This is the set of critical points of the Euclidean distance problem as can be seen from the paper [17]. △

Example 3.13. Let $X = \mathbb{V}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 - x_0^2)$ be a projective ellipsoid, and $H_\infty = \mathbb{V}(x_0)$ the hyperplane at infinity, exactly as in example 3.6. Let $a_1 = \cdots = a_n = 0$ then the metric quadric q defines the $Q = Q_{ED}$. In the affine chart $Y = X \cap \mathbb{A}^n$ is given by $Y = \mathbb{V}(c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 - 1)$ which is codimension 1. Then by the Example 3.12 the second affine reciprocal polar variety is defined by

$$\left\langle c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 - 1 \right\rangle + \left\langle 2 \times 2 \text{ minors } \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2c_1 x_1 & 2c_2 x_2 & 2c_3 x_3 \end{pmatrix} \right\rangle$$

Once the minors are calculated and factored out, we see that the affine reciprocal polar variety are the points satisfying the following polynomial system

$$\begin{aligned} c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2 &= 1 \\ x_1 x_2 (c_2 - c_1) &= 0 \\ x_1 x_3 (c_3 - c_1) &= 0 \\ x_2 x_3 (c_3 - c_2) &= 0. \end{aligned}$$

Assuming $c_1 \neq c_2 \neq c_3$, we get $W_{H_\infty^\perp}(Y) = \mathbb{V}(c_1x_1^2 + c_2x_2^2 + c_3x_3^2 - 1, x_1x_2, x_1x_3, x_2x_3)$. These are precisely the 6 points in the Figure 20. Notice that these are the same algebraic conditions we derived manually in Example 3.6 for the case $i = 2$. $\triangle$

Remark 3.9. Note that since we have chosen $a_1 = a_2 = a_3 = 0$ in the previous example. If it was also the case that $c_1 = c_2 = c_3 = 1$ then X would be a sphere like in the Example 3.5. All points on the sphere have the same distance to the origin. Therefore every point would be a critical point; hence, it is why $c_1 = c_2 = c_3 = 1$ is a degenerate example.

4 Higher-Order Classical and Reciprocal Polar Varieties

Our setting is a smooth algebraic variety X over the complex numbers $\mathbb{C}$ of dimension m. We follow [29] and [31]; however, we keep the approach as coordinate-based and computable as possible. Our goal in this section is to introduce higher-order polar varieties. The main emphasis will be on the geometry of osculating spaces, which will allow us to extend the notion of higher-order Euclidean distance optimization to higher-order polyhedral optimization, which will be given in Section 6.

In the previous section, we considered projective varieties $X \subseteq \mathbb{P}^n$ given by defining equations $\mathbb{V}(F_1, \dots, F_s)$. In this section, we shift our perspective and consider our projective variety X via an embedding $f : X \to \mathbb{P}^n$. If X is already embedded in $\mathbb{P}^n$, this map f can just be the inclusion map $i : X \hookrightarrow \mathbb{P}^n$. We will continue to assume that X is smooth and irreducible. Since f is an embedding, we will identify the point p with its image $f(p)$, and the variety X with its image $f(X)$.

Let $p \in X$. Since morphism to projective space need not be represented globally by regular functions, we will work with local coordinates. Therefore around p there is an open subset $U \subseteq X$ such that in local coordinates $t = (t_1, \dots, t_m)$ the morphism f restricted to U will be given by

$$f(t) = [\sigma_0(t) : \cdots : \sigma_n(t)],$$

where σ_i are regular functions $\mathcal{O}_X(U)$ that do not vanish simultaneously.

In many examples, X admits a global parametrization via $f : \mathbb{P}^m \to \mathbb{P}^n$. In this setting, for coordinates $(x_0, \dots, x_m)$ of $\mathbb{P}^m$, the morphism f is given by $[F_0 : \cdots : F_n]$ where F_i are homogeneous polynomials of the same degree d that do not vanish simultaneously. On the affine chart $x_0 \neq 0$, with coordinates $t_i = \frac{x_i}{x_0}$, the local representative functions $\sigma_i(t_1, \dots, t_m) = F_i(1, t_1, \dots, t_m)$. We denote $t = (t_1, \dots, t_m)$.

Definition 4.1 (Jacobian Matrix)**.** Let $p \in X$ then $f : X \to \mathbb{P}^n$ is locally given by $f(t_1, \dots, t_m) = [\sigma_0(t) : \cdots : \sigma_n(t)]$ near p. The Jacobian matrix of f evaluated at p, denoted $\mathcal{J}_{f,p}$, is the $m \times (n+1)$

matrix of first-order partial derivatives:

$$\mathcal{J}_{f,p} = \left[\frac{\partial \sigma_j}{\partial t_i}(p)\right]_{\substack{1\le i\le m\\ 0\le j\le n}}$$

Example 4.1. Let $f : \mathbb{P}^2 \to \mathbb{P}^5$ be $f([x_0 : x_1 : x_2]) = [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]$. Then on the affine chart $x_0 \neq 0$ for local coordinates $t_1 = x_1/x_0$ and $t_2 = x_2/x_0$, the morphism f restricts to $f(1, t_1, t_2) = [1 : t_1 : t_2 : t_1^2 : t_1t_2 : t_2^2]$. Since $X = \mathbb{P}^2$ is dimension $m = 2$ the Jacobian matrix $\mathcal{J}_{f,p}$, has dimension 2×6. The partial derivatives $\frac{\partial f}{\partial t_1} = (0, 1, 0, 2t_1, t_2, 0)$, $\frac{\partial f}{\partial t_2} = (0, 0, 1, 0, t_1, 2t_2)$ implies that at a point $p = (p_1, p_2)$ in this affine chart, the Jacobian matrix is

$$\mathcal{J}_{f,p} = \begin{pmatrix} 0 & 1 & 0 & 2p_1 & p_2 & 0 \\ 0 & 0 & 1 & 0 & p_1 & 2p_2 \end{pmatrix}.$$

△

Definition 4.2 (Projective Tangent Space)**.** If X is given by an embedding $f : X \hookrightarrow \mathbb{P}^n$ where locally around $p \in X$, f is represented by $f(t) = [\sigma_0(t) : \cdots : \sigma_n(t)]$, then projective tangent space at smooth point $p \in X$, denoted $T_p(f)$ is defined to be

$$\mathbb{P}\left(\operatorname{row}\begin{pmatrix} \sigma_0(p) & \dots & \sigma_n(p) \\ & \mathcal{J}_{f,p} & \end{pmatrix}\right) \subseteq \mathbb{P}^n.$$

Example 4.2. Continuing the previous example, since $f(p) = [1 : p_1 : p_2 : p_1^2 : p_1p_2 : p_2^2]$ the projective tangent space is

$$\mathbb{P}\left(\operatorname{row}\begin{pmatrix} 1 & p_1 & p_2 & p_1^2 & p_1p_2 & p_2^2 \\ 0 & 1 & 0 & 2p_1 & p_2 & 0 \\ 0 & 0 & 1 & 0 & p_1 & 2p_2 \end{pmatrix}\right).$$

△

Remark 4.1. Note that in the last two examples, the morphism f was a map $\mathbb{P}^2 \to \mathbb{P}^5$ given by homogeneous polynomials. We could have performed the same computation without passing to the local coordinates and computed the tangent space directly in homogeneous coordinates. This remark will become clearer in the following sections.

The main objects of this section are osculating spaces, which form a particular flag in projective space, whose first term is the projective tangent space. Therefore, using the higher terms of the flag, we can generalize the constructions of polar varieties by replacing the tangent space with higher-order osculating spaces and define higher-order polar varieties.

Formally, osculating spaces are defined using jet maps and the bundle of principal parts

$\mathcal{P}^k(\mathcal{O}_X(1))$. However, because our goal is to keep these constructions concrete and computable, we will take a coordinate-based approach. Instead of using sheaf-theoretic machinery, we define osculating spaces directly via higher-order partial derivatives. By taking the Taylor expansion of our parametrization up to order k, we can construct "jet matrices" whose row spans explicitly compute the osculating space at any given point. We develop this matrix-based construction in the following subsections.

4.1 Morphisms to Projective Varieties

We briefly recall how morphisms to projective space are defined globally via line bundles.

Lemma 4.1. *[22, Lemma 7.5.14] Let X be a scheme over an algebraically closed field. There is a one-to one correspondence morphisms to projective space with data of line bundle $\mathcal{L}$ with set of global sections $\{\sigma_0, \dots, \sigma_n\}$ that are base point free up to some isomorphism:*

$$\{ \text{ morphisms } f : X \to \mathbb{P}^n\} \longleftrightarrow \left\{ \begin{array}{l} \text{line bundles } \mathcal{L} \text{ on } X \text{ together with global} \\ \text{sections } \sigma_0, \dots, \sigma_n \in \Gamma(X, \mathcal{L}) \text{ such that:} \\ \text{for all } P \in X \text{ there is some } \sigma_i \text{ with } \sigma_i(P) \neq 0 \end{array} \right\}$$

For $\mathbb{P}^n$, we have a line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ whose global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ are the linear forms $\mathbb{C}[x_0, \dots, x_n]_1$ which are generated by the standard homogeneous coordinates $\{x_0, \dots, x_n\}$. The one to one correspondence in the Lemma 4.1 is realized by the pullback f^* of the global sections of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ to X, that is $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and the pullback of the basis elements $\sigma_i := f^*(x_i) \in \Gamma(X, \mathcal{L})$.

Evaluating the morphism f at any point $p \in X$ is precisely the evaluation of these $n+1$ global sections:

$$f(p) = [\sigma_0(p) : \cdots : \sigma_n(p)].$$

Because the sections are base point free by the Lemma 4.1, there is no point p such that $\sigma_i(p) = 0$ for all i. This makes $f(p)$ a well-defined point in $\mathbb{P}^n$.

Example 4.3. Let $f : \mathbb{P}^1 \to \mathbb{P}^3$ where $[u : v] \mapsto [F_0(u, v) : \cdots : F_3(u, v)]$, where F_i are all homogeneous polynomials in variables u, v in degree d not all vanishing at the same u, v except at the origin. The map f induces homomorphism f^* of graded rings

$$\begin{aligned} f^* : \mathbf{k}[x_0, x_1, x_2, x_3] &\to \mathbf{k}[u, v] \\ x_i &\mapsto F_i(u, v) \end{aligned}$$

We need to have 4 sections in $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Choosing them x_i and line bundle $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^3}(1)$, the sections will now be $f^*x_i = F_i(u, v)$. We claim that $f^*\mathcal{O}_{\mathbb{P}^3}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(d)$, which can be seen as follows:

Let $W \subseteq \mathbb{P}^3$ be an open subset, a section $s \in \Gamma(W, \mathcal{O}_{\mathbb{P}^3}(1))$ is a rational function $s = f/g$ where f and g are homogeneous polynomials where $g \neq 0$ on W and $\deg(f) - \deg(g) = 1$. If we precompose s with f, that is

$$f^*s(u,v) = \frac{f(F_0(u,v), F_1(u,v), F_2(u,v), F_3(u,v))}{g(F_0(u,v), F_1(u,v), F_2(u,v), F_3(u,v))}$$

the degree of f^*s is now $d \cdot \deg(f) - d \cdot \deg(g) = d \cdot (\deg(f) - \deg(g)) = d$ therefore it maps the sections to degree d. Then f is characterized by $(\mathcal{O}_{\mathbb{P}^1}(d), F_0(u,v), F_1(u,v), F_2(u,v), F_3(u,v))$ as we see from the map f as well. △

Example 4.4. [22, Example 3.3.11] Let $X = \mathbb{V}(x_0x_2 - x_1^2)$ and consider the map

$$f : X \to \mathbb{P}^1, \quad (x_0 : x_1 : x_2) \mapsto \begin{cases} (x_1 : x_2) & \text{if } (x_0 : x_1 : x_2) \neq (1:0:0) \\ (x_0 : x_2) & \text{if } (x_0 : x_1 : x_2) \neq (0:0:1) \end{cases}.$$

This is well defined because whenever $(x_1 : x_2)$ and $(x_0 : x_1)$ are well defined, since they are equal by the equation $x_0x_2 - x_1^2 = 0$. The inverse is given by $f^{-1} : \mathbb{P}^1 \to X$, $(y_0 : y_1) \to (y_0^2 : y_0y_1 : y_1^2)$ which is again a morphism since the components are all degree 2 homogeneous polynomials. Therefore f is an embedding. Note that this is an example where f is an embedding of projective variety that does not have a global description by homogeneous polynomials. △

4.2 Bundle of Principal Parts and Jet Maps

To define osculating spaces, we need to capture the behavior of the higher order local behavior of the map f. In calculus this would be done using Taylor expansions, however in algebraic geometry this is formalized by the bundle of principal parts.

Although the bundle of principal parts can be defined more generally for any quasi-coherent sheaf on a $\mathbf{k}$-scheme X, for the purpose of this section and onward, we are only interested in the local behavior of its fibers. For interested reader, we refer to [19, Chapter 7.2] to more details. However, for the sake of completeness, we are going to give the full description and explain the local behavior immediately after.

Definition 4.3. [19, Chapter 7.2] Let $\mathcal{L}$ be a quasi-coherent sheaf on a $\mathbf{k}$-scheme X, and write $\pi_1, \pi_2 : X \times X \to X$ for the projections onto the two factors. Let $\mathcal{I}$ be the ideal of the diagonal in $X \times X$. We set

$$\mathcal{P}^k(\mathcal{L}) := \pi_{2*}\left(\pi_1^*\mathcal{L} \otimes \mathcal{O}_{X\times X}/\mathcal{I}^{k+1}\right),$$

which is again a quasi-coherent sheaf on X. When X is smooth we call this the bundle of principal parts.

Theorem 4.2. *[19, Theorem 7.2] If $p \in X$ is a closed point, then there is a canonical identification of the fiber $\mathcal{P}^k(\mathcal{L}) \otimes \kappa(p)$ of $\mathcal{P}^k(\mathcal{L})$ at p with the sections of the restriction of $\mathcal{L}$ to the k-th-order neighborhood of p; that is,*

$$\mathcal{P}^k(\mathcal{L}) \otimes \kappa(p) = H^0\left(\mathcal{L} \otimes \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{k+1}\right)$$

as vector spaces over $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p} = \mathbf{k}$. In other words,

$$\mathcal{P}^k(\mathcal{L}) \otimes \kappa(p) = \frac{\{\textit{ germs of sections of } \mathcal{L} \textit{ at } p\}}{\{\textit{ germs vanishing to order } \geq k+1 \textit{ at } p\}}.$$

For the rest of the text, our quasi-coherent sheaf $\mathcal{L}$ will always be the line bundle $\mathcal{O}_X(1) = f^*\mathcal{O}_{\mathbb{P}^n}(1)$, and for the point $p \in X$, we will denote the fiber of $\mathcal{P}^m(\mathcal{O}_X(1))$ at point p as $\mathcal{P}_p^m(\mathcal{O}_X(1))$. By the Theorem 4.2, this is a vector space of local sections truncated at order $k+1$:

$$\mathcal{P}_p^k(\mathcal{O}_X(1)) := H^0\left(X, \mathcal{O}_X(1) \otimes \frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p^{k+1}}\right) \cong \mathbb{C}^{\binom{m+k}{k}}$$

where $\mathfrak{m}_p \subset \mathcal{O}_{X,p}$ is the maximal ideal corresponding to the point $p \in X$. The quotient by $\mathfrak{m}_p^{k+1}$ is the operation of truncating the Taylor series at order k. The dimension of this vector space corresponds to the number of monomials of total degree at most k in m variables which is precisely $\binom{m+k}{k}$.

Definition 4.4 (k-th Jet Map)**.** Let $p \in X$, then locally around p write σ in terms of local coordinates $(t_1, \dots, t_m)$. The k-th jet of a section σ at p, denoted $j_{k,p}(\sigma)$, is the $\binom{m+k}{k}$-tuple

$$j_{k,p}(\sigma) := \left(\frac{1}{|\alpha|!}\frac{\partial^{|\alpha|}\sigma}{\partial t^\alpha}(p)\right)_{\substack{\alpha \in \mathbb{N}^m \\ 0 \leq |\alpha| \leq k}} \in \mathcal{P}_p^k\left(\mathcal{O}_X(1)\right) := H^0\left(X, \mathcal{O}_X(1) \otimes \frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p^{k+1}}\right) \cong \mathbb{C}^{\binom{m+k}{k}},$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_m$, and $\frac{\partial^{|\alpha|}}{\partial t^\alpha}$ denotes the mixed partial derivative.

Example 4.5 (First Jet Map)**.** Let $p \in X$ and a global section $\sigma \in H^0\left(X, \mathcal{O}_X(1)\right)$. The first jet of σ at p is the $(m+1)$-tuple:

$$j_{1,p}(\sigma) := \left(\sigma(p), \frac{\partial \sigma}{\partial t_1}(p), \dots, \frac{\partial \sigma}{\partial t_m}(p)\right) \in H^0\left(X, \mathcal{O}_X(1) \otimes \frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p^2}\right) \cong \mathbb{C}^{m+1}.$$

Therefore the first jet map $j_{1,p}(\sigma)$ encodes the value of $\sigma(p)$ and all of its first partial derivatives. $\triangle$

For each point $p \in X$ we have a map $j_{k,p} : V \to \mathcal{P}_p^k(\mathcal{O}_X(1))$. Gluing them together yields a morphism of vector bundles on X.

$$j_k : V \otimes \mathcal{O}_X \to \mathcal{P}^k(\mathcal{O}_X(1)),$$

where $V \subseteq \Gamma(X, \mathcal{O}_X(1))$ is the vector space spanned by the sections $\sigma_0, \dots, \sigma_n$ defining our map f.

Definition 4.5 (k-th order jet matrix)**.** Given global sections $\sigma_0, \dots, \sigma_n$, we define the k-th order jet matrix as

$$A_p^{(k)}(f) := \left(\frac{1}{|\alpha|!} \frac{\partial^{|\alpha|}\sigma_\beta}{\partial t^\alpha}(p) \right)_{\substack{\alpha \in \mathbb{N}^m \\ 0 \le |\alpha| \le k \\ 0 \le \beta \le n}} = \left(j_{k,p}(\sigma_0)^\top \middle| j_{k,p}(\sigma_1)^\top \middle| \cdots \mid j_{k,p}(\sigma_n)^\top \right)$$

Example 4.6. Let $f : \mathbb{P}^2 \to \mathbb{P}^5$ be $f([x_0 : x_1 : x_2]) = [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]$ as before, therefore in the affine chart $x_0 \neq 0$ the map is locally $f(1, t_1, t_2) = (\sigma_0, \sigma_1, \dots, \sigma_5) = [1 : t_1 : t_2 : t_1^2 : t_1t_2 : t_2^2]$. For $k = 0$ since $|\alpha| = 0$ we are not taking any derivatives therefore the 0-th order jet matrix is $A_p^{(0)}(f) = (\sigma_0(p), \dots, \sigma_n(p)) = (1, p_1, p_2, p_1^2, p_1p_2, p_2^2)$. Now for $k = 1$ jet matrix will have the last rows to be $\frac{\partial f}{\partial t_1}(p)$ and $\frac{\partial f}{\partial t_2}(p)$. Therefore

$$A_p^{(1)}(f) = \begin{pmatrix} 1 & p_1 & p_2 & p_1^2 & p_1p_2 & p_2^2 \\ 0 & 1 & 0 & 2p_1 & p_2 & 0 \\ 0 & 0 & 1 & 0 & p_1 & 2p_2 \end{pmatrix}.$$

Note that this is the same matrix as in the calculation of the projective tangent space. Finally for $k = 2$, we need to calculate the second order partial derivatives as well, which are

$$\frac{1}{2!}\frac{\partial^2 f}{\partial t_1^2} = (0,0,0,1,0,0), \quad \frac{\partial^2 f}{\partial t_1 \partial t_2} = (0,0,0,0,1,0), \quad \frac{1}{2!}\frac{\partial^2 f}{\partial t_2^2} = (0,0,0,0,0,1).$$

Therefore the 2-nd order jet matrix is

$$A_p^{(2)}(f) = \begin{pmatrix} 1 & p_1 & p_2 & p_1^2 & p_1p_2 & p_2^2 \\ 0 & 1 & 0 & 2p_1 & p_2 & 0 \\ 0 & 0 & 1 & 0 & p_1 & 2p_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

As the row span of the 1-st order jet matrix was giving us the projective tangent space, the 2-nd order jet matrix gives us the second order osculating space. △

4.3 Osculating Spaces

Definition 4.6 (Osculating Space)**.** For a fixed integer k and a point p, the osculating space $\mathbb{T}_p^k(f)$ is defined as the projective span of all partial derivatives of the components of f at p up to order k, that is the row span of k-th order jet matrix at p.

$$\mathbb{T}_p^k(f) = \mathbb{P}(\mathrm{row}(A_p^{(k)}(f))).$$

Remark 4.2 (First osculating space is the tangent space)*.* Note for $k = 1$, the first jet matrix $A_p^{(1)}(f)$ is obtained by the value $f(p)$, and the Jacobian $\mathcal{J}_{f,p}$. Thus the first osculating space $\mathbb{T}_p^k(f)$ coincides with the tangent space $T_p(f)$.

Example 4.7 (Twisted Cubic Curve)**.** Let $f : \mathbb{P}^1 \to \mathbb{P}^3$ be the map defined by $(x_0 : x_1) \mapsto (x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3)$. Let $p = (p_0 : p_1) \in \mathbb{P}^1$ such that $p_0 \neq 0$ therefore locally with coordinates $t = x_1/x_0$ the map is given by $f(t) = (\sigma_0(t), \sigma_1(t), \sigma_2(t), \sigma_3(t)) = (1, t, t^2, t^3)$. Let $k = 1$, now the jet maps are

$$\begin{aligned}
j_{1,p}(\sigma_0) &= (1, 0) \\
j_{1,p}(\sigma_1) &= (t, 1) \\
j_{1,p}(\sigma_2) &= (t^2, 2t) \\
j_{1,p}(\sigma_3) &= (t^3, 3t^2)
\end{aligned}$$

therefore the first jet matrix is

$$A_p^{(1)}(f) = \begin{pmatrix} 1 & p & p^2 & p^3 \\ 0 & 1 & 2p & 3p^2 \end{pmatrix}.$$

Therefore the first osculating space is defined to be

$$\mathbb{P}\left(\mathrm{row}\begin{pmatrix} 1 & p & p^2 & p^3 \\ 0 & 1 & 2p & 3p^2 \end{pmatrix}\right).$$

In the affine chart $x_0 \neq 0$ the curve is $t \mapsto (t, t^2, t^3)$. Therefore for the point p the point is (p, p^2, p^3) and the tangent direction is $(1, 2p, 3p^2)$. Therefore the equation of the affine line is

$$(p, p^2, p^3) + t(1, 2p, 3p^2)$$

where t is a scalar.

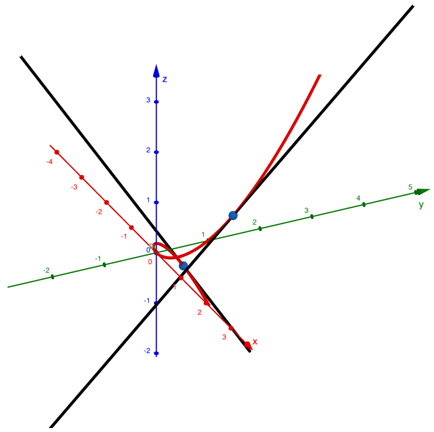

Figure 26: Twisted cubic curve with first order osculating space at $t = -1, 1$

Now let $k = 2$ then the jet maps are

$$\begin{aligned} j_{2,p}(\sigma_0) &= (1, 0, 0) \\ j_{2,p}(\sigma_1) &= (t, 1, 0) \\ j_{2,p}(\sigma_2) &= (t^2, 2t, 1) \\ j_{2,p}(\sigma_3) &= (t^3, 3t^2, 3t) \end{aligned}$$

hence the second jet matrix is

$$A_p^{(2)}(f) = \begin{pmatrix} 1 & p & p^2 & p^3 \\ 0 & 1 & 2p & 3p^2 \\ 0 & 0 & 1 & 3p \end{pmatrix}.$$

Therefore the second osculating space at the point p is $\mathbb{T}_p^2(f)$ is defined to be

$$\mathbb{P}\left(\text{row}\begin{pmatrix} 1 & p & p^2 & p^3 \\ 0 & 1 & 2p & 3p^2 \\ 0 & 0 & 1 & 3p \end{pmatrix}\right).$$

Similarly for $k = 1$ case, for the point p the point is (p, p^2, p^3) and the tangent direction is $(1, 2p, 3p^2)$ and $(0, 1, 3p)$. Therefore the parametrization of the affine plane is

$$(p, p^2, p^3) + u(1, 2p, 3p^2) + v(0, 1, 3p)$$

where u, v are scalars.

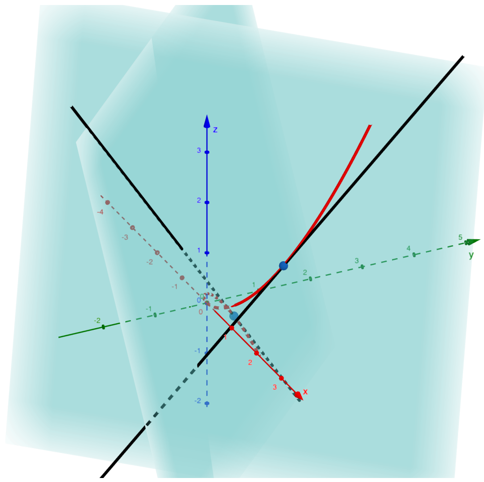

Figure 27: Twisted cubic curve with first and second order osculating space at $p = -1, 1$

Note that, we have inclusion of $\mathbb{T}_p^1(f) \subseteq \mathbb{T}_p^2(f)$, and this actually holds in general. △

Remark 4.3. Let $p \in X$, then note that we can write the k-th order jet matrix at p as

$$A_p^{(k)}(f) = \begin{pmatrix} A_p^{(k-1)}(f) \\ M \end{pmatrix}$$

for some matrix M. Therefore, we have increasing inclusions of row spans of the k-th order jet matrices at p. This implies

$$\{f(p)\} = \mathbb{T}_p^0(f) \subseteq T_p(f) = \mathbb{T}_p^1(f) \subseteq \mathbb{T}_p^2(f) \subseteq \cdots \subseteq \mathbb{T}_p^k(f) \subseteq \dots. \tag{15}$$

Remark 4.4. We now demonstrate a way to compute osculating spaces directly for morphisms of the form $f : \mathbb{P}^m \to \mathbb{P}^n$. Working in homogeneous coordinates and using the Euler's relation 2.4 for homogeneous polynomials, we show that the lower order derivatives become algebraically redundant. This allows us to define homogeneous exact-order derivative matrices, enabling us to compute the k-th osculating space simply by taking the row span of the k-th order partial derivatives of the global coordinate functions, bypassing local computation.

Example 4.8 (Twisted Cubic Curve Global)**.** Let $f : \mathbb{P}^1 \to \mathbb{P}^3$ defined by $(x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3)$. The map f induces graded algebra morphism

$$\begin{aligned} f^* : \mathbb{C}[x_0, x_1, x_2, x_3] &\to \mathbb{C}[t_0, t_1] \\ x_0 &\mapsto \sigma_0(t_0, t_1) = t_0^3 \\ x_1 &\mapsto \sigma_1(t_0, t_1) = t_0^2 t_1 \\ x_2 &\mapsto \sigma_2(t_0, t_1) = t_0 t_1^2 \\ x_3 &\mapsto \sigma_3(t_0, t_1) = t_1^3 \end{aligned}$$

For $k=1$ we will take the derivatives with respect to t_0 and t_1 hence for the point p we have

$$\begin{aligned}
j_{1,p}(\sigma_0) &= (t_0^3 : 3t_0^2 : 0)\\
j_{1,p}(\sigma_1) &= (t_0^2 t_1 : 2t_0t_1 : t_0^2)\\
j_{1,p}(\sigma_2) &= (t_0t_1^2 : t_1^2 : 2t_0t_1)\\
j_{1,p}(\sigma_3) &= (t_1^3 : 0 : 3t_1^2)
\end{aligned}$$

Therefore we will write the first osculating space $\mathbb{T}_p^1(f)$ to be row span of the jet matrix

$$A_p^{(1)}(f) = \begin{pmatrix} | & | & | & | \\ j_{1,p}(\sigma_0) & j_{1,p}(\sigma_1) & j_{1,p}(\sigma_2) & j_{1,p}(\sigma_3) \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} p_0^3 & p_0^2p_1 & p_0p_1^2 & p_1^3 \\ 3p_0^2 & 2p_0p_1 & p_1^2 & 0 \\ 0 & p_0^2 & 2p_0p_1 & 3p_1^2 \end{pmatrix}.$$

Note that row span of $A_p^{(1)}(f)$ is the same as the row span of the last 2 rows by Euler's relation on homogeneous polynomials. This motivates us to define the last 2 rows of the matrix $A_p^{(1)}(f)$ as $D_p^{(1)}(f)$ where we take only the first partial derivatives of the map f. Therefore $\operatorname{row} A_p^{(1)}(f) = \operatorname{row} D_p^{(1)}(f)$. Furthermore when we evaluate $p_0 = 1$ and $p_1 = p$ as in the previous example the matrix for $D_p^{(1)}(f)$ is:

$$\begin{pmatrix} 3 & 2p & p^2 & 0 \\ 0 & 1 & 2p & 3p^2 \end{pmatrix}$$

Note that this matrix is row equivalent to

$$\begin{pmatrix} 1 & p & p^2 & p^3 \\ 0 & 1 & 2p & 3p^2 \end{pmatrix}$$

since the second rows are the same and for the first row by row operations, specifically by relation

$$\left(1, p, p^2, p^3\right) = \frac{1}{3}\left(3, 2p, p^2, 0\right) + \frac{p}{3}\left(0, 1, 2p, 3p^2\right)$$

we get the same $\mathbb{T}_p^1(f)$ as in the previous example.

Now let $k=2$ then the jet maps are

$$\begin{aligned}
j_{2,p}(\sigma_0) &= (t_0^3 : 3t_0^2 : 0 : 3t_0 : 0 : 0)\\
j_{2,p}(\sigma_1) &= (t_0^2t_1 : 2t_0t_1 : t_0^2 : t_1 : t_0 : 0)\\
j_{2,p}(\sigma_2) &= (t_0t_1^2 : t_1^2 : 2t_0t_1 : 0 : t_1 : t_0)\\
j_{2,p}(\sigma_3) &= (t_1^3 : 0 : 3t_1^2 : 0 : 0 : 3t_1)
\end{aligned}$$

therefore arranging the jet maps in a matrix

$$A_p^{(2)}(f) = \begin{pmatrix} | & | & | & | \\ j_{2,p}(\sigma_0) & j_{2,p}(\sigma_1) & j_{2,p}(\sigma_2) & j_{2,p}(\sigma_3) \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} p_0^3 & p_0^2 p_1 & p_0 p_1^2 & p_1^3 \\ 3p_0^2 & 2p_0p_1 & p_1^2 & 0 \\ 0 & p_0^2 & 2p_0p_1 & 3p_1^2 \\ 3p_0 & p_1 & 0 & 0 \\ 0 & p_0 & p_1 & 0 \\ 0 & 0 & p_0 & 3p_1 \end{pmatrix}.$$

Note that row span of last three rows of $A_p^{(2)}(f)$ spans whole row span of $A_p^{(2)}(p)$ by Euler's relation on homogeneous polynomials. Therefore If we write last 3 row of the matrix $A_p^{(2)}(f)$ as $D_p^{(2)}(f)$ where we take only the second partial derivatives of the map f. We get $\operatorname{row} A_p^{(2)}(f) = \operatorname{row} D_p^{(2)}(f)$. Furthermore when we evaluate $p_0 = 1$ and $p_1 = p$ as in the previous example the matrix for $D_p^{(2)}(f)$ is:

$$\begin{pmatrix} 3 & p & 0 & 0 \\ 0 & 1 & p & 0 \\ 0 & 0 & 1 & 3p \end{pmatrix}$$

note that this is row equivalent to

$$\begin{pmatrix} 1 & p & p^2 & p^3 \\ 0 & 1 & 2p & 3p^2 \\ 0 & 0 & 1 & 3p \end{pmatrix}$$

which is the matrix $A_p^{(2)}(f)$ in the previous example. Therefore we got the same second order osculating space $\mathbb{T}_p^2(f)$. △

Now, in order to capture the use case of the $D_p^{(1)}(f)$ and $D_p^{(2)}(f)$ in the example, we restrict to morphisms defined globally by homogeneous polynomials. Suppose $f : \mathbb{P}^m \to \mathbb{P}^n$ is given by homogeneous polynomials $\sigma_0, \ldots, \sigma_n$ of degree d in coordinates $(x_0 : \cdots : x_m)$.

Because we are taking k-th order derivatives, we define the set of indices

$$\Lambda_k := \left\{\alpha \in \mathbb{N}^{m+1} \mid |\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_m = k\right\}.$$

and note that $|\Lambda_k| = \binom{m+k}{k}$.

Definition 4.7 (k-th order derivative matrix)**.** Let $p \in X$ be a point. We define the k-th order derivative matrix of f at p, denoted $D_p^{(k)}(f)$, as the $\binom{m+k}{k} \times (n+1)$ matrix whose rows are indexed by $\alpha \in \Lambda_k$ and whose columns are indexed by $i \in \{0, \ldots, n\}$, where

$$D_p^{(k)}(f) := \left[\frac{\partial^{|\alpha|}\sigma_i}{\partial x^\alpha}(p)\right]_{\alpha,i}.$$

Equivalently, we can write the matrix row by row as the k-th order partial derivatives of $f(x) = [\sigma_0(x) : \ldots : \sigma_n(x)]$, that is

$$D_p^{(k)}(f) = \begin{pmatrix} - & \frac{\partial^k}{\partial x^{\alpha_1}} f(p) & - \\ - & \frac{\partial^k}{\partial x^{\alpha_2}} f(p) & - \\ & \vdots & \\ - & \frac{\partial^k}{\partial x^{\alpha_N}} f(p) & - \end{pmatrix},$$

where $\alpha_1, \ldots, \alpha_N$ is the lexicographically ordered elements of Λ_k, where $N = \binom{m+k}{k}$. We will denote $D_p^{(0)}(f) := f(p)$.

In this global homogeneous setting, we have the following facts.

Proposition 4.3. *Let $k < d$ be a natural number; then the row span of the k-th derivative matrix is inside the row span of the $(k+1)$-th derivative matrix, that is*

$$\operatorname{row} D_p^{(k)}(f) \subseteq \operatorname{row} D_p^{(k+1)}(f)$$

Proof. We will show that any row of $D_p^{(k)}(f)$ can be written as a $\mathbf{k}$-linear combination of the rows of $D_p^{(k+1)}(f)$.

We first note that partial derivatives of a homogeneous polynomials are still homogeneous polynomials; therefore, we can use the Euler's relation on higher partial derivatives as well.

Let $\alpha \in \Lambda_k$ be arbitrary, then $\frac{\partial^k}{\partial x^\alpha} f(p)$ is one of the row of $D_p^{(k)}(f)$. We note that $\frac{\partial^k}{\partial x^\alpha} f$ not evaluated at p is homogeneous of degree $d - k$, therefore by Euler's relation we have

$$(d-k)\left(\frac{\partial^k}{\partial x^\alpha} f\right) = \sum_{i=0}^{m} x_i \frac{\partial}{\partial x_i}\left(\frac{\partial^k}{\partial x^\alpha} f\right).$$

When evaluated at p and dividing both sides by $(d-k)$ we see that $\frac{\partial^k}{\partial x^\alpha} f(p)$ is a $\mathbf{k}$-linear combination of

$$\left(\frac{\partial}{\partial x_i}\frac{\partial^k}{\partial x^\alpha} f\right)(p) = \frac{\partial^{k+1}}{\partial x^{\alpha+e_i}} f(p)$$

where e_i is the vector with 1 in i-th term and 0 otherwise. We see that $\alpha + e_i \in \Lambda_{k+1}$ for each i, hence any row of $D_p^{(k)}(f)$ can be written as a $\mathbf{k}$-linear combination of the rows of $D_p^{(k+1)}(f)$. ∎

Corollary 4.4. *The row span of the jet matrix is equal to the row span of the k-th order derivative matrix, that is* $\operatorname{row} A_p^{(k)} = \operatorname{row} D_p^{(k)}$.

Corollary 4.5. *The k-th osculating space is equal to projectivization of the row span of the k-th order derivative matrix,* $\mathbb{T}_p^k(f) = \mathbb{P}(\operatorname{row} D_p^{(k)})$.

Example 4.9. Let $f : \mathbb{P}^1 \to \mathbb{P}^3$ as $f = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$, where $\sigma_0 = t_0^3$, $\sigma_1 = t_0^2 t_1$, $\sigma_2 = t_0 t_1^2$, $\sigma_3 = t_1^3$, then the jet matrix will be

$$A_p^{(2)}(f) = \begin{pmatrix} t_0^3 & t_0^2 t_1 & t_0 t_1^2 & t_1^3 \\ 3t_0^2 & 2t_0 t_1 & t_1^2 & 0 \\ 0 & t_0^2 & 2t_0 t_1 & 3t_1^2 \\ 6t_0 & 2t_1 & 0 & 0 \\ 0 & 2t_0 & 2t_1 & 0 \\ 0 & 0 & 2t_0 & 6t_1 \end{pmatrix} = \begin{pmatrix} f \\ \frac{\partial f}{\partial t_0} \\ \frac{\partial f}{\partial t_1} \\ \frac{\partial f}{\partial t_0 t_0} \\ \frac{\partial f}{\partial t_0 t_1} \\ \frac{\partial f}{\partial t_1 t_1} \end{pmatrix} = \begin{pmatrix} D_p^0(f) \\ D_p^1(f) \\ D_p^2(f) \end{pmatrix}$$

by Corollary $\mathbb{T}_p^2(f) = \mathbb{P}(\operatorname{row} A_p^{(2)}(f)) = \mathbb{P}(\operatorname{row} D_p^{(2)}(f))$. △

Remark 4.5. We note that, if f is an embedding then the tangent space has constant dimension however for $k \geq 2$ the dimension of the osculating spaces $\mathbb{T}_p^k(f)$ may vary. This motivates us to define the generic dimension of the osculating space.

Definition 4.8. Denote the generic k-osculating dimension of f as $m_k = \operatorname{rk} A_p^{(k)}(f) - 1$. for generic $p \in X$. Write the open subset $X_k^\circ = \{p \in X \mid \operatorname{rk} A_p^{(k)} = m_k + 1\}$ the points where generic osculating dimension is satisfied.

Remark 4.6. We note that $m_1 = m$ where m is the dimension of the variety X. Also note that $X_1^\circ = X_{sm}$, where X_{sm} is the set of smooth points of X, therefore, in our case we have $X_{sm} = X$.

Definition 4.9. We say that X is globally k-osculating if $X_k^\circ = X$.

Remark 4.7. At $k = 1$, this corresponds to X being smooth by the previous remarks.

Definition 4.10. We say that map $f : X \to \mathbb{P}^n$ is k-regular if $X_k^\circ = X$ and if vector bundle morphism k-th jet map $j_k : V \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{P}^k(\mathcal{O}_X(1))$ is surjective.

Remark 4.8. Since vector bundles are a subcategory of $\mathcal{O}_X$-modules we have j_k surjective if and only if $j_{k,p}$ is surjective for all $p \in X$ if and only if

$$m_k + 1 = \dim(\operatorname{Im} j_{k,p}) = \dim \mathcal{P}_p^k(\mathcal{O}_X(1)) = \binom{m+k}{k}.$$

Therefore under the assumption of k-regularity the dimension of k-osculating space is $m_k = \binom{m+k}{k} - 1$.

Proposition 4.6. *[31, Proposition 2.5] Let $f : X \to \mathbb{P}^n$ be globally k-osculating for some $k \geq 1$. Then it is ℓ-osculating for all $\ell \leq k$. Similarly if f is k-regular, then it is ℓ osculating for $\ell \leq k$.*

Proof. We prove by contrapositive. Suppose f is not ℓ-osculating, by remark, we have seen that

$$A_p^{(k)}(f) = \begin{pmatrix} A_p^{(\ell)}(f) \\ M \end{pmatrix},$$

for some matrix M, which implies $A_p^{(k)}(f)$ would not be maximal rank hence f is not k-osculating. ∎

Example 4.10 (Closed Embedding that is not globally 2-osculating)**.** Write $f : \mathbb{P}^1 \to \mathbb{P}^3$ which is $(t_0^4 : t_0^3 t_1 : t_0 t_1^3 : t_1^4)$. Therefore in the affine chart $t_0 = 1$ the map f is defined to be $f(t) = (1, t, t^3, t^4)$. In the affine chart

$$A_p^{(2)}(f) = \begin{pmatrix} 1 & t & t^3 & t^4 \\ 0 & 1 & 3t^2 & 4t^3 \\ 0 & 0 & 3t & 6t^2 \end{pmatrix}.$$

We note that rank of the matrix $A_p^{(2)}(f)$ is 3 when $t \neq 0$ and 2 when $t = 0$. We can write the right kernel of the matrix $A_p^{(2)}(f)$ at $t \neq 0$ as $\operatorname{span}\{(-t^4, 2t^3, -2t, 1)^T\}$. But at $t = 0$ the matrix becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in which right kernel is given by $\operatorname{span}\{(0,0,1,0)^T, (0,0,0,1)^T\}$. So we see that dimension of $\mathbb{T}_p^2(f)$ changes according to $t = 0$. In the affine chart $t_0 = 1$ it is a curve given by (t, t^3, t^4), the second osculating space at each $t \neq 0$ will be a plane moving across the curve. However at $t = 0$, the plane crashes to a line.

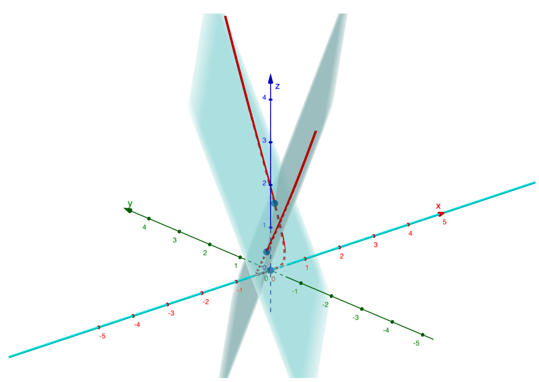

Figure 28: Three osculating spaces at the points $(1,1,1), (0,0,0), (-1,-1,1)$

In the Figure 28, the two turqoise planes are the second osculating spaces at the points $(1,1,1)$ and $(-1,-1,1)$. The turqoise line however is the second osculating space at the origin $(0,0,0)$. The red line is the curve (t, t^3, t^4). △

Example 4.11 (2-osculating that is not 2-regular)**.** Recall the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ which was given by the degree 1 combination of all monomials in coordinate rings of $\mathbb{P}^n$ and $\mathbb{P}^m$. If we change the degrees it will be a different map. Write $\mathcal{O}(d_1, d_2)$ to be the line bundle corresponding to $\mathbb{P}^n \times \mathbb{P}^m$.Now we take the degree d_1 elements of the coordinate ring of $\mathbb{P}^n$ and degree d_2 elements of the coordinate ring of $\mathbb{P}^m$. When these monomials multiplied, the corresponding structure is the vector space $\operatorname{Sym}^{d_1} V_1 \otimes \operatorname{Sym}^{d_2} V_2$ where $\mathbb{P}^n = \mathbb{P}(V_1)$ and $\mathbb{P}^m = \mathbb{P}(V_2)$. For example $\mathcal{O}(1, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ would correspond to the map

$$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5.$$
$$((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0^2 : x_0y_0y_1 : x_0y_1^2 : x_1y_0^2 : x_1y_0y_1 : x_1y_1^2)$$

Now on the affine chart $x_0 = 1$ and $y_0 = 1$ and denoting $x_1 = x$ and $y_1 = y$ the map would correspond to

$$\mathbb{A}^2 \to \mathbb{A}^6 \quad \text{where} \quad (x, y) \mapsto (1, y, y^2, x, xy, xy^2).$$

Therefore we have the embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}(\operatorname{Sym}^1 V_1 \otimes \operatorname{Sym}^2 V_2)$$

where the basis of $\operatorname{Sym}^1 V_1$ is $\{x_0, x_1\}$ and the basis of $\operatorname{Sym}^2 V_2$ is $\{y_0^2, y_0y_1, y_1^2\}$. The 2-regular condition was $X_2^\circ = X$ and $m_2 + 1 = \binom{m+2}{2} = 6$. We see that $A_p^{(2)}(f)$ cannot be rank 6 since at f_{xx} we have the zero vector, that is

$$\begin{pmatrix} 1 & y & y^2 & x & xy & xy^2 \\ 0 & 1 & 2y & 0 & x & 2xy \\ 0 & 0 & 1 & 0 & 0 & x \\ 0 & 0 & 0 & 1 & y & y^2 \\ 0 & 0 & 0 & 0 & 1 & 2y \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

However we see that rank is always 5 hence $X_2^\circ = X$ therefore it is 2-osculating. We note that the reason this example works is due to [31, Corollary 2.10] in the paper, which states that Segre-Veronese embedding is k regular if and only if $k \leq \min(\mathbf{d}) = \min(d_1, d_2)$. In our example, we have $2 > \min\{1, 2\} = 1$ therefore it is not 2 regular. △

We note that the only osculating spaces we can visualize that are nontrivial and interesting are curves in three dimensional space. Now we classify them up to degree three in an affine picture.

Example 4.12 (Degree 1 Curve)**.** A degree one curve in $\mathbb{A}^3$ is going to be given generically as a $\mathbb{A}^1 \to \mathbb{A}^3$ in which $t \mapsto (a_1 + b_1t, a_2 + b_2t, a_3 + b_3t)$ where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are generic coefficients. If we homogenize this parametrized curve we get a map

$$\begin{aligned} f : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ (t_0 : t_1) &\mapsto (t_0 : a_1t_0 + b_1t_1 : a_2t_0 + b_2t_1 : a_3t_0 + b_3t_1) \end{aligned}$$

As each of the component of f is degree 1 and f is only zero if $t_0 = t_1 = 0$, we have a well defined morphism.

Since this is a global representation of f, by discussion we had on Euler's relation and the derivative matrix we get

$$\mathbb{T}^1_{t'}(f) = \mathbb{P}(\mathrm{row}(A^{(1)}_{t'}(f))) = \mathbb{P}(\mathrm{row}(D^{(1)}_{t'}(f))) = \mathbb{P}\left(\mathrm{row}\begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}\right).$$

Since the coefficient vector $\vec{b} \neq 0$ as otherwise it would not be a degree one curve, we see that rank of $D^{(1)}_{t'}(f)$ is equal to 2. Now we find the defining equation for the $\mathbb{T}^1_{t'}(f)$ by finding the right kernel of of the matrix $D^{(1)}_{t'}(f)$ as

$$\mathrm{span}\left\{\begin{pmatrix} a_1b_2 - a_2b_1 \\ -b_2 \\ b_1 \\ 0 \end{pmatrix}, \begin{pmatrix} a_1b_3 - a_3b_1 \\ -b_3 \\ 0 \\ b_1 \end{pmatrix}\right\}$$

Therefore we see that

$$\mathbb{T}^1_{t'}(f) = \mathbb{V}((a_1b_2 - a_2b_1)x_0 + (-b_2)x_1 + b_1x_2, (a_1b_3 - a_3b_1)x_0 + (-b_3)x_1 + b_1x_3).$$

To see this in the affine chart we set $x_0 = 1$, $x_1 = x, x_2 = y, x_3 = z$, then we get the defining equation of the first osculating space of a line as

$$\mathbb{T}^1_{t'}(f) = \mathbb{V}((a_1b_2 - a_2b_1) + (-b_2)x + b_1y, (a_1b_3 - a_3b_1) + (-b_3)x + b_1z).$$

This is the intersection of two planes in $\mathbb{A}^3$, hence we get an equation of a line. As a sanity check, we know that since line is already linear, and first osculating space corresponds to the tangent line. We should get the same line we started with. $\triangle$

Example 4.13 (Degree 2 curve)**.** A degree two curve is going to be given generically as a $\mathbb{A}^1 \to \mathbb{A}^3$ in which $t \mapsto (a_1 + b_1t + c_1t^2, a_2 + b_2t + c_2t^2, a_3 + b_3t + c_3t^2)$ where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, and $\vec{c} = (c_1, c_2, c_3)$ are generic coefficients. If we homogenize this parametrized curve we get a map

$$\begin{aligned} f : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ (t_0 : t_1) &\mapsto (t_0^2 : (a_1 t_0^2 + b_1 t_0 t_1 + c_1 t_1^2) : (a_2 t_0^2 + b_2 t_0 t_1 + c_2 t_1^2) : (a_3 t_0^2 + b_3 t_0 t_1 + c_3 t_1^2)) \end{aligned}$$

We already know that $\mathbb{T}^1_{t'}(f)$ is going to correspond to the tangent line, therefore the only interesting case is the $\mathbb{T}^2_{t'}(f)$. Similarly, as in the previous example, we only need to check the $D^{(2)}_{t'}(f)$ as the elements are already homogenized. Therefore we get

$$D^{(2)}_{t'}(f) = \begin{pmatrix} 2 & 2a_1 & 2a_2 & 2a_3 \\ 0 & b_1 & b_2 & b_3 \\ 0 & 2c_1 & 2c_2 & 2c_3 \end{pmatrix}.$$

Since we are going to take the row span of $D^{(2)}_{t'}(f)$, it is easier to work with the matrix

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \end{pmatrix}$$

instead. Note that there is no dependency of t' in the entries of the $D^{(2)}_{t'}(f)$ matrix. Suppose (x_0, x_1, x_2, x_3) is in the right kernel of this matrix. Then it must be the case that $(x_1, x_2, x_3) \perp \vec{b}$ and $(x_1, x_2, x_3) \perp \vec{c}$. Therefore

$$(x_1, x_2, x_3) = \vec{b} \times \vec{c}$$

up to some scaling factor, where $\times$ is usual cross product of two vectors. Now using the first row of $D^{(2)}_{t'}(f)$, we get the equation

$$x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

using these two equations, we see that

$$x_0 = -\vec{a} \cdot (\vec{b} \times \vec{c}) = -\det \begin{pmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{pmatrix}$$

again up to same scaling factor. Therefore the right kernel of $D^{(2)}_{t'}(f)$ is

$$\operatorname{span}\left\{\begin{pmatrix} -a_1(b_2c_3-b_3c_2)+a_2(b_1c_3-b_3c_1)-a_3(b_1c_2-b_2c_1)) \\ b_2c_3-b_3c_2 \\ b_3c_1-b_1c_3 \\ b_1c_2-b_2c_1 \end{pmatrix}\right\}.$$

Hence we can write the defining equation of the second osculating space as

$$\mathbb{V}\left(-\det\begin{pmatrix} - & \vec{a} & - \\ - & \vec{b} & - \\ - & \vec{c} & - \end{pmatrix}+(\vec{b}\times\vec{c})\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

where we set $x_0 = 1$, $x_1 = x$, $x_2 = y$, and $x_3 = z$.

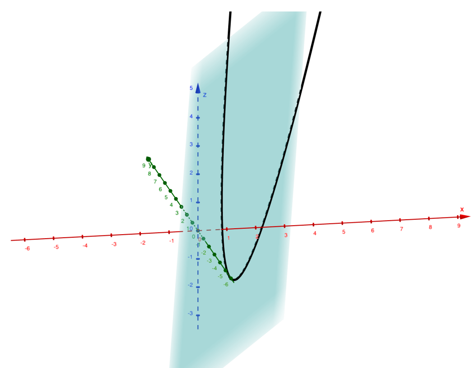

Figure 29: Osculating space of the degree 2 curve

We note the reader that since there is not t' in the defining equation it is just a fixed plane that contains the quadratic curve. △

Example 4.14 (Degree 3 curve)**.** The most interesting osculating plane where we directly see how it generalized the tangent space is when the curve is degree three and we look at the second osculating space. The reason for that is when it is a cubic curve in space, although we take 2 partial derivatives, there will still be some variables depending on the variable t. Therefore the second osculating space is also going to move on where we are on the curve.

A degree three curve is given generically as an image of a map $\mathbb{A}^1 \to \mathbb{A}^3$ with $t \mapsto (a_1 + b_1t + c_1t^2 + d_1t^3, a_2 + b_2t + c_2t^2 + d_2t^3, a_3 + b_3t + c_3t^2 + d_3t^3)$ where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$, and $\vec{d} = (d_1, d_2, d_3)$ are generic coefficients. If we homogenize this parametrized curve, we get a map $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ where

$$\begin{aligned}(t_0 : t_1) \mapsto (t_0^3 &: (a_1 t_0^3 + b_1 t_0^2 t_1 + c_1 t_0 t_1^2 + d_1 t_1^3) \\ &: (a_2 t_0^3 + b_2 t_0^2 t_1 + c_2 t_0 t_1^2 + d_2 t_1^3) \\ &: (a_3 t_0^3 + b_3 t_0^2 t_1 + c_3 t_0 t_1^2 + d_3 t_1^3)\end{aligned}$$

As before we are going to calculate the second derivative matrix $D_{t'}^{(2)}(f)$ which will be more involved as follows:

$$D_t^{(2)}(f) = \begin{pmatrix} 6t_0 & (6a_1 t_0 + 2b_1 t_1) & (6a_2 t_0 + 2b_2 t_1) & (6a_3 t_0 + 2b_3 t_1) \\ 0 & (2b_1 t_0 + 2c_1 t_1) & (2b_2 t_0 + 2c_2 t_1) & (2b_3 t_0 + 2c_3 t_1) \\ 0 & (2c_1 t_0 + 6d_1 t_1) & (2c_2 t_0 + 6d_2 t_1) & (2c_3 t_0 + 6d_3 t_1) \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

Now in order to find the right kernel of this matrix, similarly to the case of degree 2 curve we are going to use the shape of the matrix to conclude that for $x = (x_1, x_2, x_3)^T$ we must have $\vec{x} \perp r_2$ and $\vec{x} \perp r_3$ which implies $x = r_2 \times r_3$ where $\times$ denotes the cross product. Then

$$\begin{aligned}\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 2b_1 t_0 + 2c_1 t_1 \\ 2b_2 t_0 + 2c_2 t_1 \\ 2b_3 t_0 + 2c_3 t_1 \end{pmatrix} \times \begin{pmatrix} 2c_1 t_0 + 6d_1 t_1 \\ 2c_2 t_0 + 6d_2 t_1 \\ 2c_3 t_0 + 6d_3 t_1 \end{pmatrix} \\ &= \left(2t_0 \vec{b} + 2t_1 \vec{c}\right) \times (2t_0 \vec{c} + 6t_1 \vec{d}) \\ &= 4t_0^2 (\vec{b} \times \vec{c}) + 12 t_0 t_1 (\vec{b} \times \vec{d}) + 12 t_1^2 (\vec{c} \times \vec{d})\end{aligned}$$

Furthermore using the first row of the matrix and since we are assuming that $t_0 \neq 0$ for the affine picture we get

$$\begin{aligned}x_0 &= \frac{1}{6t_0}\Big[(-6a_1 t_0 - 2b_1 t_1) x_1 + (-6a_2 t_0 - 2b_2 t_1) x_2 + (-6a_3 t_0 - 2b_3 t_1) x_3\Big] \\ &= \frac{1}{6t_0}\Big[-6t_0(a_1 x_1 + a_2 x_2 + a_3 x_3) - 2t_1(b_1 x_1 + b_2 x_2 + b_3 x_3)\Big] \\ &= -(\vec{a} \cdot \vec{x}) - \frac{1}{3}\frac{t_1}{t_0}(\vec{b} \cdot \vec{x})\end{aligned}$$

Now since $\vec{x} = r_2 \times r_3$ already we get

$$x_0 = -\vec{a} \cdot \vec{x} - \frac{1}{3}\frac{t_1}{t_0} 12 t_1^2 \det \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}.$$

Now we will further assume that $\vec{a} = 0$ as it only allows us to translate the curve in the affine space. We will also assume that $t_0 = 1$ since we are interested in the picture in the affine space. Therefore we get the equation

$$x_0 = -4t_1^3 \det \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}$$

and

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4(\vec{b} \times \vec{c}) + 12t_1(\vec{b} \times \vec{d}) + 12t_1^2(\vec{c} \times \vec{d})$$

After writing $x_1 = x, x_2 = y, x_3 = z$ and $x_0 = 1$, the defining equation for second order osculating space for the affine part is

$$\mathbb{V}\left(-4t^3 \det \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} + 4\det \begin{pmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} + 12t \det \begin{pmatrix} x & y & z \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{pmatrix} + 12t^2 \det \begin{pmatrix} x & y & z \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix}\right).$$

We can further fact check that when (x, y, z) evaluated at the points of the curve, that is when

$$\begin{aligned} x &= b_1t + c_1t^2 + d_1t^3 \\ y &= b_2t + c_2t^2 + d_2t^3 \\ z &= b_3t + c_3t^2 + d_3t^3 \end{aligned}$$

the equation satisfies, and when evaluated at the tangent line equation, that is when equation is evaluated at

$$\begin{aligned} x &= (b_1t + c_1t^2 + d_1t^3) + l(b_1 + 2c_1t + 3d_1t^2) \\ y &= (b_2t + c_2t^2 + d_2t^3) + l(b_2 + 2c_2t + 3d_2t^2) \\ z &= (b_3t + c_3t^2 + d_3t^3) + l(b_3 + 2c_3t + 3d_3t^2) \end{aligned}$$

where l is arbitrary variable for parametrization of the tangent line the equation again is satisfied. $\triangle$

4.4 Higher Order Classical and Reciprocal Polar Varieties

We will now generalize the classical and reciprocal polar varieties by replacing tangent space with the higher-order osculating spaces. We are going to introduce them for completeness. For more details, we refer to [29] and for the relationship of how their degrees are related to the higher-order Euclidean distance optimization problem, we refer to [31].

Definition 4.11. [29, Section 3] Let $f : X \to \mathbb{P}^n$ be an algebraic morphism. We define higher-order polar variety with respect to $i \in \{0, \ldots, \dim X = m\}$ to be

$$P_{k,i}(f,V) := \overline{\left\{x \in X_k^\circ \mid \dim\left(\mathbb{T}_p^k(f) \cap V\right) > m_k - \operatorname{codim}(V)\right\}}$$

where $V \subseteq \mathbb{P}^n$ is a linear subspace of dimension $n - m_k + i - 2$.

Remark 4.9. See that when $k = 1$, we have $\mathbb{T}_p^1(f) = T_p(f)$ and the dimension $m_1 = m$. Hence, for $k = 1$ we get exactly the i-th polar variety $P_i(X,V)$. Therefore the interesting examples of higher-order polar varieties occur for $k > 1$.

Example 4.15 (Twisted Cubic Curve)**.** Let $k = 2$ and let the projective variety $X \subseteq \mathbb{P}^3$ to be the twisted cubic curve, therefore dimension is $m = 1$. It is also given by the image of the morphism $f : \mathbb{P}^1 \to \mathbb{P}^3$ where $[t_0 : t_1] \mapsto [t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3]$. We have already seen that this is 2-regular global morphism, therefore $m_2 = \binom{3}{2} - 1 = 2$. Furthermore by definition $\mathbb{T}_p^2(f)$ is defined by the projectivization of the row span of 2nd order derivative matrix

$$D_p^{(2)}(f) = \begin{pmatrix} 6t_0 & 2t_1 & 0 & 0 \\ 0 & 2t_0 & 2t_1 & 0 \\ 0 & 0 & 2t_0 & 6t_1 \end{pmatrix}.$$

Let $i = 1 = \dim X$, and consider a generic projective subspace $V \subseteq \mathbb{P}^3$, then higher-order polar variety $P_{2,1}(f,V)$ is given by the equation

$$\begin{aligned} P_{2,1}(f,V) &= \overline{\left\{x \in X_2^\circ \mid \dim\left(\mathbb{T}_p^2(f) \cap V\right) > -1\right\}} \\ &= \left\{p \in X \mid \dim\left(\mathbb{T}_p^2(f) \cap V\right) \geq 0\right\} \\ &= \left\{p \in X \mid \mathbb{T}_p^2(f) \cap V \neq \emptyset\right\} \\ &= \left\{p \in X \mid L \subseteq \mathbb{T}_p^2(f)\right\} \end{aligned}$$

since $m_2 - \operatorname{codim}(V) = 2 - (2 - 1 + 2) = -1$, and since $\mathbb{T}_p^2(f)$ is a hyperplane. The dimension of V is $\dim V = 3 - 2 + 1 - 2 = 0$ therefore V is a point $v \in \mathbb{P}^3$. Write the point v as $v = [v_0 : v_1 : v_2 : v_3] \in \mathbb{P}^3$. Now the condition of $V \subseteq \mathbb{T}_p^2(f)$ in the higher-order polar variety is the condition of

$v \in row(D_p^{(2)}(f))$. Since the $D_p^{(2)}$ is already full rank we can write it as a rank condition

$$\left\langle 4 \times 4 \text{ minors of } \begin{pmatrix} v_0 & v_1 & v_2 & v_3 \\ 6t_0 & 2t_1 & 0 & 0 \\ 0 & 2t_0 & 2t_1 & 0 \\ 0 & 0 & 2t_0 & 6t_1 \end{pmatrix} \right\rangle .$$

Therefore the second order polar variety $P_{2,1}(f, V)$ is given by the images by f of solutions $[t_0 : t_1]$ satisfying the determinant

$$\det \begin{pmatrix} v_0 & v_1 & v_2 & v_3 \\ 6t_0 & 2t_1 & 0 & 0 \\ 0 & 2t_0 & 2t_1 & 0 \\ 0 & 0 & 2t_0 & 6t_1 \end{pmatrix} = 0.$$

Consider the example where $v = [1 : 0 : -1 : 0]$, then the determinant evaluates to

$$24t_1^3 - 72t_0^2 t_1 = 24t_1(t_1^2 - 3t_0^2) = 24t_1(t_1 - \sqrt{3}t_0)(t_1 + \sqrt{3}t_0) = 0.$$

Therefore the solution are $\{[1 : 0], \left[1 : \sqrt{3}\right], [1 : -\sqrt{3}]\}$. When we map these points to $\mathbb{P}^3$ using the morphism f we get three points in the twisted cubic curve

$$\left\{[1 : 0 : 0 : 0], [1 : \sqrt{3} : 3 : 3\sqrt{3}], [1 : -\sqrt{3} : 3 : -3\sqrt{3}]\right\}.$$

In this example all of these points are happened to be in the chart $x_0 \neq 0$ therefore we can visualize it in the $\mathbb{A}^3$ chart.

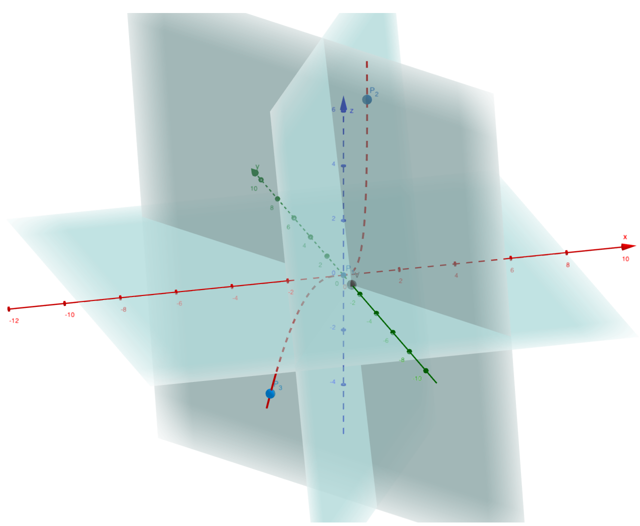

Figure 30: $P_{2,1}(f, V)$ in the affine chart $\mathbb{A}^3$

Here in the Figure 30, the three turquoise planes are the three osculating spaces of three points

P_0, P_1, P_2. The grey point is the point v, and finally the polar variety is the three points. $\triangle$

Similar to the higher-order polar varieties, we are going to generalize the reciprocal polar varieties by changing the normal space into higher-order normal space.

Fix a nonsingular quadric hypersurface $Q = \mathbb{V}(q) \subseteq \mathbb{P}^n$. In Section 3.1 we have defined the polarity. We are going to use the same notation.

In order to define the higher-order reciprocal polar varieties, we need to define higher-order normal spaces.

Definition 4.12. [31, Definition 4.1] Let $f : X \to \mathbb{P}^n$ be a morphism, then for $p \in X_k^\circ$ the k-th order normal space of (f, Q) at p is

$$\mathbb{N}_p^k(f) := \left(\mathbb{T}_p^k(f)\right)^\perp + f(p) \subseteq \mathbb{P}^n.$$

Remark 4.10. For $k = 1$ since $\mathbb{T}_p^1(f) = T_p(f)$ we get the usual normal space $N_p(f)$.

Remark 4.11. [31, Remark 4.2] By the sequence (15) of higher-order osculating spaces and by Lemma 3.4 we have the inclusions of normal spaces analogous to (15)

$$\{p\} \subseteq \mathbb{N}_p^k(f, Q) \subseteq \cdots \subseteq \mathbb{N}_p^2(f, Q) \subseteq \mathbb{N}_p(f, Q) \subseteq \mathbb{P}^n.$$

Definition 4.13 (Reciprocal Polar Variety)**.** Let $k \geq 1$ and $i \in \{0, \ldots, \dim X = m\}$. Let $K \subseteq \mathbb{P}^n$ be a linear subspace of codimension $n - m_k + i$. Then the i-th reciprocal polar variety of order k is

$$W_k^\perp(X, K) = \overline{\{p \in X_k^\circ \mid \mathbb{N}_p^k(f, Q) \cap K \neq \emptyset\}}.$$

5 Degree

In this chapter, we will be working on connecting the degree of classical polar varieties and reciprocal polar varieties to optimization questions. Since we will be using intersection theory, the first chapter is devoted to recalling the facts that will be used in subsequent chapters.

5.1 Intersection Theory

Intersection theory is the branch of algebraic geometry that studies how subvarieties intersects. When they intersect in finitely many points, we are interested in counting them. Intersection theory makes this counting problem systematic way.

We will use the classical textbooks [19, 21] for the theorems and for the facts in intersection theory. The goal of this section is to introduce a ring $A^*(X)$ called the Chow ring of X and for any subvariety $Y \subseteq X$, we associate its class $[Y] \in A^*(X)$ generalizing the degree of a curve.

As always, assume X is a smooth projective variety in $\mathbb{P}^n$ of dimension m and codimension c.

Definition 5.1. The group of cycles on X, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties of X. It is graded by dimension, therefore we write $Z_k(X)$ for the group of cycles that are formal linear combinations of subvarieties of dimension k. Similarly we write $Z^{m-k}(X)$ for the group of cycles that are formal linear combinations of subvarieties of codimension $m-k$. Therefore we can write $Z(X) = \bigoplus_{i=0}^{m} Z_i(X) = \bigoplus_{i=0}^{m} Z^{m-i}(X)$.

Definition 5.2. [19, Definition 1.2] Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\langle \Phi \cap (\{t_A\} \times X) \rangle - \langle \Phi \cap (\{t_B\} \times X) \rangle,$$

where $t_A, t_B \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $\{t\} \times X$. We say that two cycles are rationally equivalent if their difference is in $\mathrm{Rat}(X)$, and we say that two subvarieties are rationally equivalent if their associated cycles are rationally equivalent.

Example 5.1. Let $X = \mathbb{P}^2$ with homogeneous coordinates $[x_0 : x_1 : x_2]$ and $\mathbb{P}^1$ with homogeneous coordinates $[t_0 : t_1]$. we will show that $U = \mathbb{V}(x_1^2 + x_2^2 - x_0^2)$ and $V = \mathbb{V}(x_1^2 - x_2^2)$ are rationally equivalent. Let $\mathbb{P}^1$ homogeneous coordinates $(t_0 : t_1)$. Define $\Phi = \mathbb{V}(F) \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ given by $F(t_0, t_1, x_0, x_1, x_2) = t_0(x_1^2 + x_2^2 - x_0^2) - t_1(x_1^2 - x_2^2)$. Note that Φ is not contained in any fiber $\{t\} \times X$. Now we evaluate the cycles associated to the slices at $t_A = [1 : 0]$ and $t_B = [0 : -1]$.

At $t_A = [1 : 0]$, we substitute $t_0 = 1, t_1 = 0$ into F to get

$$F(t_A, x) = 1(x_1^2 + x_2^2 - x_0^2) - 0(x_1^2 - x_2^2) = x_1^2 + x_2^2 - x_0^2.$$

The cycle of this intersection is our first subvariety:

$$\langle \Phi \cap \left(\{t_A\} \times \mathbb{P}^2\right) \rangle = [U].$$

At $t_B = [0 : -1]$, we substitute $t_0 = 0, t_1 = -1$ into F to get

$$F(t_B, x) = 0(x_1^2 + x_2^2 - x_0^2) + 1(x_1^2 - x_2^2) = (x_1^2 - x_2^2).$$

The cycle of this intersection is our second subvariety:

$$\langle \Phi \cap \left(\{t_B\} \times \mathbb{P}^2\right) \rangle = [V].$$

Therefore $[U] - [V] \in \mathrm{Rat}(X)$ hence $U \sim V$. △

Example 5.2 (Rational Equivalence of Composite Cycles)**.** Let $X = \mathbb{P}^2$ with homogeneous coordinates $[x_0 : x_1 : x_2]$.

Consider the 1-cycle U consisting of the line $x_0 = 0$ with multiplicity 2, and the line $x_1 = 0$

with multiplicity 1

$$U = 2[\mathbb{V}(x_0)] + 1[\mathbb{V}(x_1)].$$

Consider another 1-cycle V consisting of the line $x_2 = 0$ with multiplicity 2, and the line $x_0 - x_1 = 0$ with multiplicity 1

$$V = 2[\mathbb{V}(x_2)] + 1[\mathbb{V}(x_0 - x_1)].$$

We wish to prove that $U \sim V$ by showing $[U] - [V] \in \mathrm{Rat}(\mathbb{P}^2)$. We are going to define a set of irreducible subvarieties $\Phi_i \subset \mathbb{P}^1 \times \mathbb{P}^2$ and their combination of the differences will be $[U] - [V]$.

Let $\mathbb{P}^1$ have coordinates $[t_0 : t_1]$ and $\Phi_1 = \mathbb{V}(F_1) \subset \mathbb{P}^1 \times \mathbb{P}^2$ with $F_1(t, x) = t_0 x_0 - t_1 x_2$. Evaluating at the points at $t_A = [1 : 0]$ and $t_B = [0 : -1]$ we get

$$\begin{aligned}\langle \Phi_1 \cap (\{t_A\} \times \mathbb{P}^2) \rangle &= [\mathbb{V}(x_0)] \\ \langle \Phi_1 \cap (\{t_B\} \times \mathbb{P}^2) \rangle &= [\mathbb{V}(x_2)].\end{aligned}$$

Thus, the difference $[\mathbb{V}(x_0)] - [\mathbb{V}(x_2)]$ is a generator of $\mathrm{Rat}(\mathbb{P}^2)$.

Let $\Phi_2 = \mathbb{V}(F_2) \subset \mathbb{P}^1 \times \mathbb{P}^2$ with $F_2(t, x) = t_0 x_1 - t_1(x_0 - x_1)$. Evaluating at the points $t_A = [1 : 0]$ and $t_B = [0 : -1]$ we get

$$\begin{aligned}\langle \Phi_2 \cap (\{t_A\} \times \mathbb{P}^2) \rangle &= [\mathbb{V}(x_1)] \\ \langle \Phi_2 \cap (\{t_B\} \times \mathbb{P}^2) \rangle &= [\mathbb{V}(x_0 - x_1)].\end{aligned}$$

Thus, the difference $[\mathbb{V}(x_1)] - [\mathbb{V}(x_0 - x_1)]$ is another generator of $\mathrm{Rat}(\mathbb{P}^2)$.

Because $\mathrm{Rat}(\mathbb{P}^2)$ is defined as the subgroup generated by these differences, any integer linear combination of these generators must also be inside in the subgroup. We construct the linear combination

$$\begin{aligned}&2\left(\langle \Phi_1 \cap \{t_A\} \rangle - \langle \Phi_1 \cap \{t_B\} \rangle\right) + 1\left(\langle \Phi_2 \cap \{t_A\} \rangle - \langle \Phi_2 \cap \{t_B\} \rangle\right) \\ &= 2\left([\mathbb{V}(x_0)] - [\mathbb{V}(x_2)]\right) + 1\left([\mathbb{V}(x_1)] - [\mathbb{V}(x_0 - x_1)]\right) \\ &= \left(2[\mathbb{V}(x_0)] + 1[\mathbb{V}(x_1)]\right) - \left(2[\mathbb{V}(x_2)] + 1[\mathbb{V}(x_0 - x_1)]\right) \\ &= U - V,\end{aligned}$$

showing that $U - V$ is an element of $\mathrm{Rat}(\mathbb{P}^2)$, hence $U \sim V$. △

Definition 5.3. [19, Definition 1.3] The Chow group of X is the quotient

$$A(X) = Z(X)/\,\mathrm{Rat}(X),$$

the group of rational equivalence classes of cycles on X. If $Y \in Z(X)$ is a cycle, we write

$[Y] \in A(X)$ for its equivalence class; if $Y \subset X$ is a subvariety, we abuse notation slightly and denote simply by $[Y]$ the class of the cycle $\langle Y \rangle$ associated to Y.

We say that two subvarieties A, B of X intersect transversely at a point p if A, B and X are all smooth at p and

$$\operatorname{codim}\left(T_p A \cap T_p B\right)=\operatorname{codim}\left(T_p A\right)+\operatorname{codim}\left(T_p B\right).$$

Furthermore we say that the subvarieties $A, B \subset X$ are generically transverse if they meet transversely at a general point of each component C of $A \cap B$.

Two cycles $A=\sum n_i A_i$ and $B=\sum m_j B_j$ are generically transverse if each A_i is generically transverse to each B_j.

Theorem 5.1. *[19, Theorem 1.5] If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying the condition:*

If two subvarieties A, B of X are generically transverse, then $[A][B]=[A \cap B]$. This structure makes Chow group

$$A(X)=\bigoplus_{c=0}^{\dim X} A^c(X)$$

into an associative, commutative ring, graded by codimension, called the Chow ring of X.

The proof relies on the Moving Lemma, which is stated below and proved in [19].

Theorem 5.2. *[19, Theorem 1.6] Let X be a smooth quasi-projective variety, then*

1. *For every $\alpha, \beta \in A(X)$ there are generically transverse cycles $A, B \in Z(X)$ with $[A]=\alpha$ and $[B]=\beta$.*

2. *The class $[A \cap B]$ is independent of the choice of such cycles A and B.*

Remark 5.1. For any variety X we can express the group $\operatorname{Rat}(X)$ in terms of divisor classes as well. The book [21] uses this approach to define rational equivalences. When the underlying variety X is smooth two definitions of rational equivalances collide therefore we have chosen the definition given in [19]. For interested readers we refer to discussion of these equivalences to [19, 1.3.2 Rational equivalence via divisors] and [21, 1.6 Alternate Definition of Rational Equivalence].

Since our varieties X that we are interested in is inside of the projective space $\mathbb{P}^n$ we are going to recall some facts about their chow rings.

Theorem 5.3. *[19, Theorem 2.1] The Chow ring of $\mathbb{P}^n$ is*

$$A\left(\mathbb{P}^n\right)=\mathbb{Z}[H] /\left(H^{n+1}\right),$$

where $H \in A^1(\mathbb{P}^n)$ is the rational equivalence class of a hyperplane; more generally, the class of a variety of codimension k and degree d is dH^k.

Theorem 5.4. *[19, Theorem 2.10] The Chow ring of $\mathbb{P}^r \times \mathbb{P}^s$ is given by the formula*

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong A(\mathbb{P}^r) \otimes A(\mathbb{P}^s).$$

Equivalently, if $h, h' \in A^1(\mathbb{P}^r \times \mathbb{P}^s)$ denote the pullbacks, via the projection maps, of the hyperplane classes on $\mathbb{P}^r$ and $\mathbb{P}^s$ respectively, then

$$A(\mathbb{P}^r \times \mathbb{P}^s) \cong \mathbb{Z}[h, h'] / \left(h^{r+1}, (h')^{s+1}\right).$$

Moreover, the class of the hypersurface defined by a bihomogeneous form of bidegree (d, e) on $\mathbb{P}^r \times \mathbb{P}^s$ is $dh + eh'$.

For the definition of degree we refer to [19, 1.3.6 Functoriality] and [21, 1.4 Push-forward of Cycles].

Definition 5.4 (Degree)**.** Let X be smooth projective variety over an algebraically closed field. Let $\alpha = \sum n_i[p_i] \in Z_0(X)$ be a 0-cycle. The degree of α denoted either as deg or $\int_X$ is the group homomorphism $Z_0(X) \to \mathbb{Z}$ defined as

$$\deg(\alpha) = \int_X \alpha = \sum n_i.$$

This map is also well defined for the rational equivalences [21, Theorem 1.4], giving a well defined homomorphism on chow group

$$\int_X : A_0(X) \to \mathbb{Z}$$

Remark 5.2. The definition of degree above is given in [21, Definition 1.4], however in our case since X is smooth projective variety there are some things we have omitted while defining degree. First of all structure morphism $\pi : X \to \operatorname{Spec}(\mathbf{k})$ is proper since X is projective. Therefore pushforward π_* is well defined, and since the map $\int_X$ is pushforward $\pi_* : A_0(X) \to A_0(\operatorname{Spec}\mathbf{k}) \simeq \mathbb{Z}$ we have a well defined degree. Furthermore since we are working in algebraically closed field $\kappa(p_i) \simeq \mathbf{k}$ therefore field extension $[\kappa(p_i) : \mathbf{k}] = 1$.

Definition 5.5. Let $X \subset \mathbb{P}^n$ be a smooth projective variety of dimension m, and let $H \in A^1(X)$ denote the hyperplane class of X, given by the restriction of the standard hyperplane class of $\mathbb{P}^n$ to X. For any irreducible subvariety $V \subset X$ of dimension k, the degree of V is the intersection number:

$$\deg(V) = \int_X [V] \cdot H^k$$

where $H^k \in A^k(X)$ is the k-fold intersection product of H with itself.

Example 5.3 (Linear subspace of dimension k)**.** Consider $X = \mathbb{P}^n$ and the subvariety V be the k dimensional linear subspace. By Theorem 5.3 we have $[V] = H^{n-k}$ where H is the class of a hyperplane. Therefore

$$\deg(V) = \int_{\mathbb{P}^n} [V] \cdot H^k = \int_{\mathbb{P}^n} H^{n-k} H^k = \int_{\mathbb{P}^n} H^n = 1.$$

Since $H^n \in A^n(\mathbb{P}^n) \simeq A_0(\mathbb{P}^n)$ represents a single point. △

Example 5.4 (Hypersurface of degree d)**.** Let $X = \mathbb{P}^n$ and let $V = \mathbb{V}(F)$ be a smooth subvariety given by F a homogeneous polynomial of degree d. By Theorem 5.3 $[V] = dH \in A^1(\mathbb{P}^n)$. Therefore

$$\deg(V) = \int_{\mathbb{P}^n} [V] \cdot H^{n-1} = \int_{\mathbb{P}^n} dH \cdot H^{n-1} = d \int_{\mathbb{P}^n} H^n = d,$$

as expected. △

Theorem 5.5. *[19, Theorem 1.23] Let $f : Y \to X$ be a map of smooth quasi-projective varieties.*

- *There is a unique map of groups $f^* : A^c(X) \to A^c(Y)$ such that whenever $A \subset X$ is a subvariety generically transverse to f we have*

  $$f^*([A]) = \left[f^{-1}(A)\right].$$

  This equality is also true without the hypothesis of generic transversality as long as $\operatorname{codim}_Y\left(f^{-1}(A)\right) = \operatorname{codim}_X(A)$ *and A is Cohen-Macaulay. The map also induces graded ring homomorphism $f^* : A(X) \to A(Y)$.*

- *(Projection formula) The map $f_* : A(Y) \to A(X)$ is a map of graded modules over the graded ring $A(X)$. More explicitly, if $\alpha \in A^k(X)$ and $\beta \in A_l(Y)$, then*

  $$f_*\left(f^*\alpha \cdot \beta\right) = \alpha \cdot f_*\beta \in A_{l-k}(X).$$

Remark 5.3. [21, Definition 1.4] For X and Y proper schemes, which is always the case in our setting when X and Y are projective varieties in $\mathbb{P}^n$, for $f : X \to Y$ a proper morphism and for any cycle $\alpha \in A(X)$ we have

$$\int_X \alpha = \int_Y f_*\alpha.$$

Recall that the properness of X and Y implies structure morphisms $\pi_X : X \to \operatorname{Spec}(\mathbf{k})$ and $\pi_Y : Y \to \operatorname{Spec}(\mathbf{k})$ to be well defined and the degree maps are pushforwards, that is $\int_X = \pi_{X*}$ and $\int_Y = \pi_{Y*}$. Since X and Y are defined over $\operatorname{Spec}(\mathbf{k})$ the map $f : X \to Y$ commutes with the structure morphisms $\pi_X = \pi_Y \circ f$. Pushforward is functorial therefore

$$\pi_{X*} = \pi_{Y*} \circ f_*.$$

Hence if we evaluate the cycle $\alpha \in A(X)$ we get

$$\int_X \alpha = \pi_{X*}(\alpha) = \pi_{Y*} \circ f_*(\alpha) = \int_Y f_*(\alpha).$$

Example 5.5. [19, 4.1 Schubert cells and Schubert cycles] Let $\mathbf{a}$ be a sequence such that $n-k \geq a_0 \geq \cdots \geq a_k \geq 0$ and let $\mathcal{V}$ and $\mathcal{K}$ be two flags in $\mathbb{P}^n$. Since they differ by an element of GL_{n+1} therefore by [19, Theorem 1.7] two Schubert cycles $\Sigma_{\mathbf{a}}(\mathcal{V})$, $\Sigma_{\mathbf{a}}(\mathcal{K})$ have the same class and we are going to denote it as $\sigma_{\mathbf{a}} = [\Sigma_{\mathbf{a}}(\mathcal{V})] \in A(\mathrm{Gr}(k, \mathbb{P}^n))$. $\triangle$

Although for any subvariety V of X we can associate a class $[V]$ in the chow ring $A(X)$, we can expand this more by using so called Chern classes. Informally for any vector bundle $\mathcal{E}$ on X of rank r, we can associate Chern classes $c_i(\mathcal{E}) \in A^i(\mathcal{E})$ for $i \in \{1, \ldots, r\}$. We will get to examples on calculation with them but before that their existence and properties are given in the theorem below.

Theorem 5.6. *[19, Theorem 5.3] There is a unique way of assigning to each vector bundle $\mathcal{E}$ on a smooth quasi-projective variety X a class $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \cdots \in A(X)$ in such a way that:*

- *If $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is $1 + c_1(\mathcal{L})$, where $c_1(\mathcal{L}) \in A^1(X)$ is the class of the divisor of zeros minus the divisor of poles of any rational section of $\mathcal{L}$.*

- *If $\tau_0, \ldots, \tau_{r-i}$ are global sections of $\mathcal{E}$, and the degeneracy locus D where they are dependent has codimension i, then $c_i(\mathcal{E}) = [D] \in A^i(X)$.*

- *If*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

is a short exact sequence of vector bundles on X then

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}) \in A(X).$$

- *If $\varphi : Y \to X$ is a morphism of smooth varieties, then*

$$\varphi^*(c(\mathcal{E})) = c\left(\varphi^*(\mathcal{E})\right).$$

5.2 Polar Classes

Definition 5.6 (Polar class)**.** Given a smooth projective variety X in $\mathbb{P}^n$, the polar variety $P(X, V)$ is a subvariety of X. We define i-th polar class of X as $p_i(X) = [P(X, V)] \in A(X)$ where $i = 0, \ldots, m$ and $\dim V = c + i - 2$.

We have shown in Corollary 2.13 that $\operatorname{codim}_X(P(X,V)) = i$ for i-th polar variety therefore it also makes sense why we use the indexing $i = 0, \dots, m$.

Definition 5.7 (Polar degree)**.** Given an i-th polar class of X $p_i(X) \in A^i(X)$, the degree of the class $p_i(X)$ is called polar degree and denoted as $\mu_i(X)$.

Example 5.6 (0-th polar degree is degree of the variety)**.** Since we have already established that 0-th polar variety is the X itself in Remark 2.2, we have $p_0(X) = [P(X,V)] = [X]$ therefore the 0-th polar degree $\mu_0(X) = \deg X$. △

Now, the first theorem we are going to show is the statement we have written in the definition of classical polar variety Definition 2.2 which is for $i = 0, \dots, m$ the polar degree $\mu_i(X)$ does not depend on the subspace V for a generic V.

Theorem 5.7. *[27, Proposition 1.2] Let X be a smooth projective variety of dimension m. Let $i \in \{0, \dots, m\}$ then the polar degree $\mu_i(X)$ does not depend on the subspace V for a generic V.*

Proof. We will be using the Theorem 5.5. First of all, recall that $\gamma_X : X \to Gr(m, \mathbb{P}^n)$ is a map of smooth projective varieties. Recall that in Theorem 2.11 we have defined the polar variety $P(X,V) \subseteq X$ as the pullback of Schubert variety $\Sigma_m(V)$ through the Gauss map γ_X, that is $P(X,V) = \gamma_X^{-1}(\Sigma_m(V))$. Furthermore, let $\mathcal{V}$ be flag containing projective subspace V than we have also shown that $\Sigma_m(V) = \Sigma_{\mathbf{a}}$ where $\mathbf{a} = (1, \dots, 1, 0, \dots, 0)$ in Lemma 2.10.

We have also shown that $\operatorname{codim}_X(P(X,V)) = i$ and $\operatorname{codim}_{Gr(m,\mathbb{P}^n)}(\Sigma_{\mathbf{a}}) = \sum_{j=0}^{n} a_j = i$. It is also a fact that every Schubert variety is Cohen-Macaulay [11, Proposition 2.2.5]. Hence

$$p_i = [P(X,V)] = [\gamma_X^{-1}(\Sigma_m(V))] = [\gamma_X^{-1}(\Sigma_{\mathbf{a}})] = \gamma_X^*[\Sigma_{\mathbf{a}}] = \gamma_X^*\sigma_{\mathbf{a}}.$$

Note that polar class p_i does not depend on V since we have already established in Example 5.5 that, we have shown that the class of Schubert variety $\sigma_{\mathbf{a}} = [\Sigma_{\mathbf{a}}]$ does not depend on the flag $\mathcal{V}$. Hence $\mu_i(X)$ does not depend on the projective subspace V. ■

5.3 Polar Degrees using Multidegrees

Fix X to be a smooth projective variety in $\mathbb{P}^n$. In this section we are going to introduce multidegree $\delta_i(X)$ to connect it with polar degrees $\mu_i(X)$. Recall that in Section 2.4. We have define the dual projective space $(\mathbb{P}^n)^\vee$ as $\mathrm{Gr}(n-1, \mathbb{P}^n)$ which parametrizes hyperplanes in $\mathbb{P}^n$. The isomorphism between them was given in Section 3.1. In Theorem 5.3, we have shown the Chow ring of the projective space $\mathbb{P}^n$ is $A^*(\mathbb{P}^n) = \mathbb{Z}[H]/H^{n+1}$ where H is the class of hyperplane H in $\mathbb{P}^n$. Via the isomorphism of dual projective space $(\mathbb{P}^n)^\vee$, its Chow ring will also be $A((\mathbb{P}^n)^\vee) = \mathbb{Z}[H']/(H')^{n+1}$ where H' is the class of hyperplane in $(\mathbb{P}^n)^\vee$.

Recall that conormal variety defined in Definition 2.12 was $CN(X) \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ given by the incidence relation

$$CN(X) = \left\{(p, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid p \in X,\ T_pX \subseteq H\right\} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee .$$

Furthermore conormal variety $CN(X)$ comes with two surjective projections π_1 and π_2 given by $\pi_1 : CN(X) \to X$ and $\pi_2 : CN(X) \to X^\vee$ where $\pi_i = pr_i \circ j$ where $j : CN(X) \to \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ and pr_1 and pr_2 be the projections $\mathbb{P}^n \times (\mathbb{P}^n)^\vee \to \mathbb{P}^n$ and $\mathbb{P}^n \times (\mathbb{P}^n)^\vee \to (\mathbb{P}^n)^\vee$ respectively. We have also defined the dual variety $X^\vee$ to be the image $\pi_2(CN(X)) \subseteq (\mathbb{P}^n)^\vee$, this is the set of hyperplanes in $\mathbb{P}^n$ that is tangent to X.

We now recall some facts on dual varieties and conormal variety. For simplicity we will always assume X is smooth projective variety and that the ground field is characteristic 0 and algebraically closed.

Theorem 5.8. *[23, Biduality Theorem] For any projective variety $X \subset \mathbb{P}^n$, we have $(X^\vee)^\vee = X$. Moreover, if p is a smooth point of X and H is a smooth point of $X^\vee$, then H is tangent to X at p if and only if p, regarded as a hyperplane in $(\mathbb{P}^n)^\vee$, is tangent to $X^\vee$ at H.*

Theorem 5.9. *[19, Theorem 10.20 (Reflexivity)] If $X \subset \mathbb{P}^n$ is any projective variety and $X^\vee \subset (\mathbb{P}^n)^\vee$ its dual, then the conormal variety $CN(X) \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ is equal to $CN(X^\vee) \subset (\mathbb{P}^n)^\vee \times \mathbb{P}^n$ with the factors reversed. It follows that $(X^\vee)^\vee = X$, that is the dual of the dual of X is X.*

Theorem 5.10. *[9, Proposition 2.4] Let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety of dimension* $\dim X = m$ *then* $\dim CN(X) = n - 1$ *in particular* $\operatorname{codim} CN(X) = n + 1$.

Proof. First, note that hyperplane $H \in (\mathbb{P}^n)^\vee$ also be represented by the vector $u = [u_0 : \cdots : u_n] \in \mathbb{P}^n$, so that $H = \{p \in \mathbb{P}^n \mid \sum_{i=0}^n u_i p_i = 0\}$. See that actually $H = u^\perp$, where $\perp$ is the polarity defined by the quadric $Q = Q_{ED}$. Define the normal space as $N_pX = T_pX^\perp$, then we can also represent the conormal variety as $CN(X) = \{(p, u) \in \mathbb{P}^n \times \mathbb{P}^n \mid p \in X,\ u \in N_pX\}$ since $u \in T_pX^\perp$ is equivalent to $T_pX \subseteq u^\perp = H$. We have $\dim N_pX = \dim(T_pX^\perp) = n - m - 1$ by Corollary 3.4.

With this view $CN(X)$ has the structure of projective bundle, where each fiber along point p, $\pi_1^{-1}(p)$ is the normal space $N_pX \simeq \mathbb{P}^{n-m-1}$, see section [19, 9.1 Projective bundles] for projective bundles. Finally by fiber dimension theorem, we have

$$\dim CN(X) = \dim X + \dim(\pi_1^{-1}(p)) = m + (n - m - 1) = n - 1.$$

Since $\dim(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = n + n = 2n$, we get $\operatorname{codim} CN(X) = n + 1$. ∎

Remark 5.4. Note that since the dual variety $X^\vee$ is the projection of conormal variety $CN(X)$ onto second coordinate, we have $\dim(X^\vee) \leq n - 1$.

Now take a generic hyperplane $H \in X^\vee$ tangent to X at some point. Define *contact locus* as

$$\mathrm{Cont}(H, X) := \pi_1(\pi_2^{-1}(H)) = P(X, H) \tag{16}$$

where the last equality is given in the Section 2.4 since for H hyperplane $H^\vee = H$, which follows from the Example 2.13. We call $\dim(\mathrm{Cont}(H, X))$ for a generic $H \in (\mathbb{P}^n)^\vee$ to be *dual defect* of X. Since H is a hyperplane the non-transversality condition in the polar variety becomes $T_pX \subseteq H$, therefore

$$\mathrm{Cont}(H, X) = \{p \in X \mid T_pX \subseteq H\}.$$

Note that this is a generic fiber of π_2, therefore we have the following observation:

Theorem 5.11. *We have* $\dim(X^\vee) = n - 1 - \dim(\mathrm{Cont}(H, X))$ *for generic hyperplane* $H \in X^\vee$. *In particular, dual variety* $X^\vee$ *is a hypersurface if and only if dimension of the contact locus* $\mathrm{Cont}(H, X)$ *is* 0.

Proof. We have already shown that $\dim(CN(X)) = n - 1$. Also note that for generic $H \in X^\vee$, the dimension $\dim(\mathrm{Cont}(H, X))$ is the dimension of generic fiber of π_2. Therefore we have $\dim(CN(X)) = \dim(X^\vee) + \dim(\mathrm{Cont}(H, X))$ which means $\dim(X^\vee) = n - 1 - \dim(\mathrm{Cont}(H, X))$. ∎

This motivates us to define the dual defect of X.

Definition 5.8. [24, Definition 5.1] The dual defect of X is defined to be

$$\mathrm{def}(X) = \dim(\mathrm{Cont}(H, X))$$

for a generic $H \in (\mathbb{P}^n)^\vee$. A projective variety $X \subseteq \mathbb{P}^n$ is dual defective if $\mathrm{def}(X) > 0$ otherwise it is dual non-defective.

Theorem 5.12. *[24, Corollary 2.2] We have* $CN(X) = CN(X^\vee)$ *if and only if* $\mathrm{Cont}(H, X)$ *is a projective subspace of* $\mathbb{P}^n$.

Since due to our setting reflexivity $CN(X) = CN(X^\vee)$ always holds, we have the following corollary.

Corollary 5.13. *[15, Corollary 2.6] The dual variety* $X^\vee$ *is a hypersurface if and only if the generic tangent hyperplane to* X *is tangent at a single point.*

Let H and H' be hyperplanes in $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$ and $[H]$ and $[H']$ be their classes in Chow rings $A(\mathbb{P}^n)$ and $A((\mathbb{P}^n)^\vee)$. Write

$$h = pr_1^*([H]) = [H \times (\mathbb{P}^n)^\vee], \quad h' = pr_2^*([H']) = [\mathbb{P}^n \times H']$$

. We also have $A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = \mathbb{Z}[h,h']/\left\langle h^{n+1},(h')^{n+1}\right\rangle$ by Theorem 5.4. Since $\operatorname{codim} CN(X) = n+1$ its class $[CN(X)] \in A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$ is

$$[CN(X)] = \delta_0(X)h^n(h') + \delta_1(X)h^{n-1}(h')^2 + \cdots + \delta_{n-1}(X)h(h')^n = \sum_{j=0}^{n-1}\delta_j(X)h^{n-j}(h')^{j+1}. \quad (17)$$

Definition 5.9. (Multi Degree) Let $i \in \{0,\dots,n-1\}$, the coefficients $\delta_i(X)$ of the class of the conormal variety $[CN(X)]$ in the Chow ring $A^*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$ is called the i-th multidegree of X.

Proposition 5.14. *Let $L_1 \subseteq \mathbb{P}^n$ and $L_2 \subseteq (\mathbb{P}^n)^\vee$ be two generic projective subspaces of dimension $n-i$ and $i+1$. Then the i-th multidegree $\delta_i(X)$ is the cardinality of $CN(X) \cap (L_1 \times L_2)$.*

Proof. Codimensions of L_1 and L_2 are i and $n-i-1$ respectively, so their classes are $[L_1] = [H]^i \in A(\mathbb{P}^n)$ and $[L_2] = [H']^{n-i-1} \in A((\mathbb{P}^n)^\vee)$. Then using Theorem 5.4 we have the class of $[L_1 \times L_2] = h^i(h')^{n-i-1} \in A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = \mathbb{Z}[h,h']/\left\langle h^{n+1},(h')^{n+1}\right\rangle$. The intersection becomes the product in the Chow ring

$$\begin{aligned}[CN(X) \cap (L_1 \times L_2)] &= [CN(X)][L_1 \times L_2] \\ &= \left(\sum_{j=0}^{n-1}\delta_j(X)h^{n-j}(h')^{j+1}\right)\left(h^i(h')^{n-i-1}\right) \\ &= \sum_{j=0}^{n-1}\delta_j(X)h^{n-j+i}(h')^{n+j-i}.\end{aligned}$$

Now we are going to consider the cases of the summand where the index j is bigger and smaller the i.

In the case that $j > i$ we have $(h')^{n+j-i} \in \langle (h')^{n+1}\rangle$ therefore the summand where $j > i$ will vanish in the Chow ring $A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = \mathbb{Z}[h,h']/\left\langle h^{n+1},(h')^{n+1}\right\rangle$.

In the case that $i > j$ we will have $h^{n-j+i} \in \langle h^{n+1}\rangle$, therefore the summand where $i > j$ will vanish in the Chow ring $A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = \mathbb{Z}[h,h']/\left\langle h^{n+1},(h')^{n+1}\right\rangle$.

Hence the only term in the summation that will not vanish is the index $j = i$ which yields

$$[CN(X) \cap (L_1 \times L_2)] = \delta_i h^n(h')^n.$$

Since the ambient space $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$ is dimension $2n$, the coefficient $\delta_i(X)$ is exactly the number $\#(CN(X) \cap (L_1 \times L_2))$. ∎

Theorem 5.15. *[24, Theorem 5.1] Let $X \subseteq \mathbb{P}^n$ be a irreducible projective variety of dimension m. Let r be an integer such that*

$$\delta_0(X) = \dots \delta_{r-1}(X) = 0, \delta_r(X) \neq 0$$

Then for all $i \in \{r, \ldots, m\}$,

$$\delta_i(X) > 0.$$

Theorem 5.16. *[24, Theorem 3.4] The dual variety* $X^\vee$ *is dimension* $n-1-r$ *if*

$$\delta_0(X) = \ldots \delta_{r-1}(X) = 0, \delta_r(X) \neq 0.$$

Furthermore $\delta_r(X) = \deg(X^\vee)$.

Example 5.7. Let $V \subseteq \mathbb{P}^n$ be a projective subspace of dimension k. We have $\dim V^\vee = n-1-\operatorname{def}(V)$ by the Theorem 5.11. We have already discussed in Example 2.13 that

$$V^\vee = \{H \in (\mathbb{P}^n)^\vee \mid V \subset H\}.$$

Let $H \in V^\vee$ then it is just a hyperplane in $\mathbb{P}^n$ such that $V \subset H$.

We are going to show that $\operatorname{Cont}(H, V) = V$ for a generic $H \in V^\vee$. By (16) it is immediate

$$\begin{aligned} \operatorname{Cont}(H, V) = P(V, H) &= \{\mathbf{p} \in V \mid H + \mathbf{p} \not\pitchfork T_{\mathbf{p}} V\} \\ &= \{\mathbf{p} \in V \mid H \not\pitchfork V\} \\ &= \{\mathbf{p} \in V \mid \dim(H+V) < n\} \\ &= \{\mathbf{p} \in V \mid \dim(H) = n-1 < n\} \\ &= V. \end{aligned}$$

Therefore $\dim V^\vee = n-1-k$ furthermore $\operatorname{codim} V^\vee = n-(n-1-k) = k+1$. △

Theorem 5.17. *For all* $0 \leq i \leq m$*, we have the relationship between polar degree and multi degree of* X *as follows*

$$\mu_i(X) = \delta_{m-i}(X).$$

Proof. The proof will be given similarly on how more general case was proven in the papers [31, 15].

Recall that $\mu_i(X) = \deg(P(X, V))$ for $V \subset \mathbb{P}^n$ generic projective subspace of dimension $c+i-2$. Its dual $V^\vee$ has codimension $c+i-1$ by Example 5.7. Therefore its class in the Chow ring $A((\mathbb{P}^n)^\vee) = \mathbb{Z}[H'] / \langle (H')^n \rangle$ is $[V^\vee] = [H']^{c+i-1}$. Let $j : CN(X) \hookrightarrow \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ be the inclusion map. Then

$$\mu_i(X) = \deg(P(X,V)) = \int [H]^{m-i} \cdot [P(X,V)] \tag{18}$$
$$= \int [H]^{m-i} \cdot \left[\pi_1\left(\pi_2^{-1}\left(V^\vee\right)\right)\right] \tag{19}$$
$$= \int [H]^{m-i} \cdot \pi_{1*}\left(\pi_2^*\left([V^\vee]\right)\right) \tag{20}$$
$$= \int [H]^{m-i} \cdot \pi_{1*}\left(\pi_2^*\left([H']^{c+i-1}\right)\right) \tag{21}$$
$$= \int \pi_{1*}\left(\pi_1^*\left([H]^{m-i}\right) \cdot \pi_2^*\left([H']^{c+i-1}\right)\right) \tag{22}$$
$$= \int \pi_{1*}\left(\pi_1^*([H])^{m-i} \cdot \pi_2^*\left([H']\right)^{c+i-1}\right) \tag{23}$$
$$= \int \pi_{1*}\left(j^*\left(h^{m-i}\right) \cdot j^*\left((h')^{c+i-1}\right)\right) \tag{24}$$
$$= \int \pi_{1*}\left(j^*\left(h^{m-i} \cdot (h')^{c+i-1}\right)\right) \tag{25}$$
$$= \int \pi_{1*}\left(h^{m-i} \cdot (h')^{c+i-1} \cdot [CN(X)]\right) \tag{26}$$
$$= \int h^{m-i} \cdot (h')^{c+i-1} \cdot [CN(X)] \tag{27}$$
$$= \delta_{m-i}(X) \tag{28}$$

In (18) we use the definition of degree, (19) is the Theorem 2.14 which was the characterization of polar variety in terms of conormal variety, (20) is the discussion pullbacks and push forwards in Section 5, (21) is the identification $[V^\vee] = [H']^{c+i-1}$ where codimension was shown in Example 5.7, (22) is the projection formula given in Theorem 5.5, (23) is by pullback being a ring morphism, (24) is obtained by rewriting $\pi_i = pr_i \circ j$. (25) is by pullback being a ring morphism again, (26) is by the pullback of embedding j, (27) is because degree is invariant under f_* which is the content of Remark 5.3, finally (28) is by Proposition 5.14. ■

The following observation is although due to Urabe in [38] we will use the discussion used in [24, Section 2] to prove it.

Theorem 5.18. *For reflexive X that is $CN(X) = CN(X^\vee)$, we have $\delta_j(X^\vee) = \delta_{n-1-j}(X)$*

Proof. We have discussed $A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = \mathbb{Z}[h,h']/\left\langle h^{n+1}, (h')^{n+1}\right\rangle$ where h and h' are the pullbacks of the hyperplane class in $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$ respectively to the $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$.

Similarly, let $\hat{h}$ and $\hat{h}'$ be the pullbacks of the hyperplane classes of $(\mathbb{P}^n)^\vee$ and $(\mathbb{P}^n)^{\vee\vee}$ to the $(\mathbb{P}^n)^\vee \times (\mathbb{P}^n)^{\vee\vee}$. We have an isomorphism $(\mathbb{P}^n)^\vee \times (\mathbb{P}^n)^{\vee\vee} \simeq \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ given by interchanging $\mathbb{P}^n$ with $(\mathbb{P}^n)^\vee$, and with the identification $(\mathbb{P}^n)^{\vee\vee} = \mathbb{P}^n$. Furthermore this induces an isomorphism on the Chow rings $A((\mathbb{P}^n)^\vee \times (\mathbb{P}^n)^{\vee\vee}) \simeq A(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$ given by $\hat{h} \mapsto h'$ and $\hat{h}' \mapsto h$.

We can write $[CN(X^\vee)] \in A((\mathbb{P}^n)^\vee \times (\mathbb{P}^n)^{\vee\vee})$ using the classes $\hat{h}$ and $\hat{h}'$:

$$[CN(X^\vee)] = \delta_0(X^\vee)\hat{h}^n(\hat{h}') + \delta_1(X^\vee)\hat{h}^{n-1}(\hat{h}')^2 + \cdots + \delta_{n-1}(X^\vee)\hat{h}(\hat{h}')^n.$$

With the isomorphism of Chow rings induced by the identification we can write is as

$$[CN(X^\vee)] = \delta_0(X^\vee)h(h')^n + \delta_1(X^\vee)h^2(h')^{n-1} + \cdots + \delta_{n-1}(X^\vee)h^n(h').$$

However since X is reflexive the class $[CN(X)]$ should be equal to $[CN(X^\vee)]$. Since we already have

$$[CN(X)] = \delta_0(X)h^n(h') + \delta_1(X)h^{n-1}(h')^2 + \cdots + \delta_{n-1}(X)h(h')^n$$

only possibility that they are equal is when $\delta_j(X^\vee) = \delta_{n-1-j}(X)$. ∎

Corollary 5.19. *[10, Theorem 4.15] Let $X \subset \mathbb{P}^n$ be an irreducible projective variety of dimension m, and let $\alpha(X) := m - \operatorname{def}(X)$. Then*

- $\mu_i(X) > 0$ *if and only if* $0 \leq i \leq \alpha(X)$
- $\mu_0(X) = \deg(X)$
- $\mu_{\alpha(X)}(X) = \deg(X^\vee)$
- $\mu_i(X) = \mu_{\alpha(X)-i}(X^\vee)$

Remark 5.5. Note that in [10, Theorem 4.15] their $\alpha(X)$ was given by $\alpha(X) := \dim X - \operatorname{codim} X^\vee + 1$. However by the theorems we have shown it is equal to the $\alpha(X) = m - \operatorname{def}(X)$.

Proof. We will show the $\mu_i(X) = \mu_{\alpha(X)-i}(X^\vee)$ first. Note that by Theorem 5.18 we have $\delta_j(X^\vee) = \delta_{n-1-j}(X)$, and by Theorem 5.17 we have $\delta_j(X) = \mu_{m-j}(X)$ where $m = \dim X$. Recall that $\dim X^\vee = n - 1 - \operatorname{def}(X)$, therefore we get

$$\mu_i(X) = \delta_{m-i}(X) = \delta_{n-1-m+i}(X^\vee) = \mu_{(n-1-\operatorname{def}(X))-(n-1-m+i)}(X^\vee) = \mu_{m-\operatorname{def}(X)-i}(X^\vee) = \mu_{\alpha(X)-i}(X^\vee).$$

We have shown $\mu_0(X) = \deg(X)$ in Example 5.6. The $\mu_{\alpha(X)}(X) = \deg(X^\vee)$ is implied by $\mu_i(X) = \mu_{\alpha(X)-i}(X^\vee)$ since after choosing $i = \alpha(X)$ by $\mu_i(X) = \mu_{\alpha(X)-i}(X^\vee)$ we get

$$\mu_{\alpha(X)}(X) = \mu_0(X^\vee) = \deg X^\vee.$$

So only the statement $\mu_i(X) > 0$ if and only if $0 \leq i \leq \alpha(X)$ is left to be proven. Using the identification $\delta_j(X) = \mu_{m-j}(X)$, it is equivalent to prove $\delta_i(X) > 0$ if and only if $\operatorname{def}(X) \leq i \leq m$.

Assume $\delta_i(X) > 0$, we have shown in Proposition 5.14 that $\delta_i(X)$ is the cardinality of $CN(X) \cap (L_1 \times L_2)$ where $L_1 \subseteq \mathbb{P}^n$ and $L_2 \subseteq (\mathbb{P}^n)^\vee$ are a general projective subspace of dimension $n - i$ and

$i+1$ respectively. For $CN(X) \cap (L_1 \times L_2)$ to be non-empty, the projections of $CN(X)$ onto $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$ must intersect L_1 and L_2. We will consider each projection separately.

Projection of $CN(X)$ to the first coordinate is X, and as X is an irreducible projective variety of dimension m, it intersects a general subspace of codimension i if and only if $i \leq m$. Projection of $CN(X)$ to the second coordinate is $X^\vee$ and by Theorem 5.11 we have $\dim(X^\vee) = n-1-\operatorname{def}(X)$. The variety $X^\vee$ intersects a general subspace of codimension $n-1-i$ if and only if $n-1-i \leq n-1-\operatorname{def}(X)$ equivalent to $\operatorname{def}(X) \leq i$. Therefore $\delta_i(X) > 0$ implies $\operatorname{def}(X) \leq i \leq m$. Proving the first direction.

For the converse, by 5.15 whose proof was given in [24, Theorem 5.1] we get $\operatorname{def}(X) \leq i \leq m$ implying $\delta_j(X) > 0$. ■

Corollary 5.20. *[10, Theorem 4.20] Let X be a smooth, irreducible projective variety, and let $m := \dim X$. Then,*

$$\mu_i(X) = \sum_{j=0}^{i} (-1)^j \binom{m-j+1}{m-i+1} \deg\left(c_j(X)\right)$$

Proof. Let $i \in \{0, \ldots, m\}$, then using [24, Equation (3)] we have the formula for multidegree in terms of chern classes

$$\delta_i(X) = \sum_{j=i}^{m} (-1)^{m-j} \binom{j+1}{i+1} \deg\left(c_{m-j}(X)\right).$$

By Theorem 5.17 we have the relationship $\mu_i(X) = \delta_{m-i}(X)$ therefore

$$\begin{aligned}\mu_i(X) = \delta_{m-i}(X) &= \sum_{j=m-i}^{m} (-1)^{m-j} \binom{j+1}{i+1} \deg\left(c_{m-j}(X)\right) \\ &= \sum_{j=0}^{i} (-1)^j \binom{m-j+1}{m-i+1} \deg\left(c_j(X)\right)\end{aligned}$$

hence we got the desired result. ■

5.4 Euclidean Distance Degree

In this section we are going to connect both the reciprocal polar varieties and classical polar varieties to Euclidean distance optimization problem. We are going to show that the Euclidean distance degree is equal to the sum of polar degrees. The Euclidean distance degree is the number of complex critical points of the squared distance to a general point outside the variety. For an introduction we will refer to the book [10] and for the main theorems we will be referring to the paper [17].

Before getting into the Euclidean distance optimization question, we will start off by explaining algebraic degree and optimizing a polynomial function on a real variety.

Let Y be a smooth real variety with ideal $I_Y = \langle f_1, \ldots, f_s \rangle$ with codimension c in $\mathbb{A}^n$. Let $g : Y \to \mathbb{R}$ be an objective function. The polynomial optimization question can be written as

$$\text{minimize } g(p) \text{ subject to } p \in Y. \tag{29}$$

Using Lagrange multiplier method, the critical points of g are the points $p \in Y$ such that

$$\nabla g(p) \in \operatorname{span}\{\nabla f_1(p), \ldots, \nabla f_s(p)\} = \operatorname{row}(\mathcal{J}_Y(p)). \tag{30}$$

The dimension of Y is m, therefore dimension of tangent space T_pY is also m. Since $T_pY = \ker \mathcal{J}_Y$, we have $\dim \operatorname{row}(\mathcal{J}_Y(p)) = n - m = c$. Therefore condition in (30) is equivalent to

$$\operatorname{rk} \begin{pmatrix} \nabla g(p) \\ \mathcal{J}_Y(p) \end{pmatrix} = c.$$

Therefore we can write the critical points of g as an ideal

$$I_{\text{crit}} = \langle f_1, \ldots, f_s \rangle + \left\langle (c+1) \times (c+1) \text{ minors } \begin{pmatrix} \nabla g \\ \mathcal{J}_Y \end{pmatrix} \right\rangle. \tag{31}$$

When this ideal is zero dimensional the degree of I_{crit} is called the algebraic degree of the optimization (29). If g is defined generically, this ideal will be zero dimensional.

Example 5.8. Let $g(x) = x_1 + x_2$, and let $Y = \mathbb{V}(x_1^2 + x_2^2 - 1)$ then $\nabla g = (1, 1)$ constant vector and the Jacobian is $\mathcal{J}_Y = (2x_1, 2x_2)$. Since codimension $c = 1$ the critical ideal is

$$\begin{aligned} I_{\text{crit}} &= \left\langle x_1^2 + x_2^2 - 1 \right\rangle + \left\langle 2 \times 2 \text{ minors } \begin{pmatrix} 1 & 1 \\ 2x_1 & 2x_2 \end{pmatrix} \right\rangle \\ &= \left\langle x_1^2 + x_2^2 - 1, x_2 - x_1 \right\rangle. \end{aligned}$$

Therefore the critical points are $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$. Therefore, there are 2 critical points in which both are real. In the Figure 31 the critical points are the two blue points. The black line is the equation $x + y = k$ where k is the value of the function $g(x, y) = k$. The red arrow is the vector that is perpendicular to the red circle which is the variety Y, the black arrow is the vector that is perpendicular to the black line. The blue points are exactly the points where two normal vectors are linearly dependent. $\triangle$

We are going to define the Euclidean distance optimization in $\mathbb{A}^n$. We work over field of real numbers $\mathbf{k} = \mathbb{R}$. Let $Y \subseteq \mathbb{A}^n_{\mathbb{R}}$ be a real smooth affine variety given by $\mathbb{V}(f_1, \ldots, f_s)$ for data point $u \in \mathbb{A}^n_{\mathbb{R}} \setminus Y$. We define the function $d_u : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined by $d_u(x) = \sqrt{q(x - u)}$ where $q(x) = q_{ED}(x) = x_1^2 + \cdots + x_n^2$ positive definite quadratic form in $\mathbb{R}^n$. Note that $d_u(x)^2$ is a

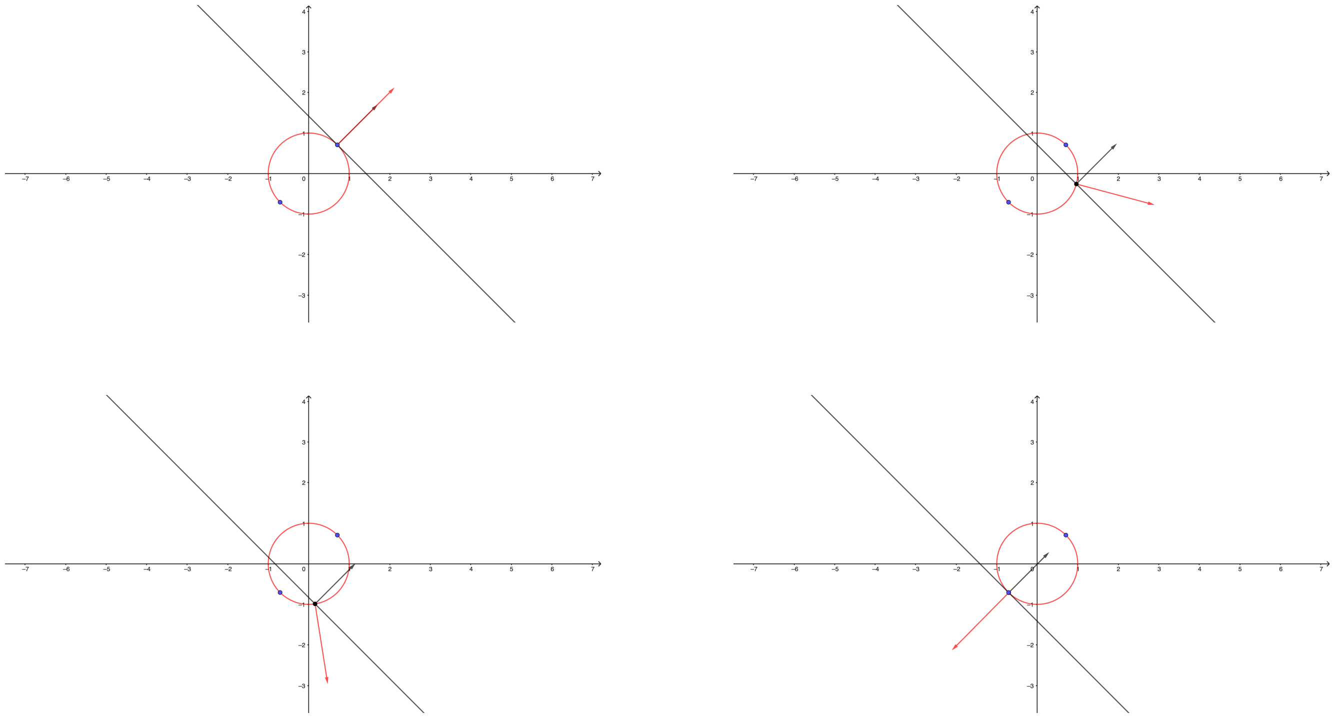

Figure 31: Visualisation of the critical points with Lagrange multipliers method.

polynomial function on $\mathbb{R}^n$. Therefore, we can do a polynomial optimization for the objective function $d_u(x)^2$

$$\text{minimize } d_u(x)^2 = \sum_{i=1}^{n}(x_i - u_i)^2 \text{ subject to } x \in Y. \tag{32}$$

Equation (32) is called Euclidean distance optimization problem for Y.

To count the complex critical points of (32) as we defined algebraic degree, we pass to the complexification. That is we view Y as $Y_{\mathbb{C}} \subseteq \mathbb{A}^n_{\mathbb{C}}$ defined by same functions $f_1, \ldots, f_s$. The objective function extends to polynomial map $d_u^2 : \mathbb{C}^n \to \mathbb{C}$. Note that quadratic form $q = x_1^2 + \cdots + x_n^2$ is a bilinear symmetric form over $\mathbb{C}^n$ however not a Hermitian inner product.

Definition 5.10 (Euclidean distance degree)**.** The number of complex critical points of (32) for a general point $u \in \mathbb{A}^n \setminus Y$ is called the Euclidean distance degree, denoted as EDdegree(Y) and it does not depend on u for sufficiently generic u.

Note that complex critical points of Euclidean distance optimization problem (32) can be solved by calculating the critical ideal denoted as $C_{Y,u}$ in Equation (31), where we change g to

$d_u(x)^2$ instead. Since $\nabla(d_u(x)^2) = (2(x_1-u_1), \dots, 2(x_n-u_n))$ the critical ideal will be exactly

$$C_{Y,u} = \langle f_1, \dots, f_s \rangle + \left\langle (c+1)\times(c+1) \text{ minors of } \begin{pmatrix} x_1-u_1 & \dots & x_n-u_n \\ - & \nabla f_1 & - \\ & \vdots & \\ - & \nabla f_s & - \end{pmatrix} \right\rangle. \tag{33}$$

Normally the ideal saturation with singular points of Y is also necessary; see for instance [17, Equation (2.1)], however since we are assuming Y to be smooth in the whole section, we dropped the saturation.

Now let $X \subseteq \mathbb{P}^n$ be the smooth projective variety then we can consider its affine cone $C(X) \subseteq \mathbb{A}^{n+1}$. Therefore we can expand the Euclidean distance problem (32) and definition of Euclidean distance degree 5.10 to projective variety X by considering optimization question (32) on $C(X)$, we will write $\text{EDdegree}(X) := \text{EDdegree}(C(X))$ for the Euclidean distance degree of a projective variety. For more details on how they differ from the affine case, we refer to [17, 2 Equations defining critical points].

We would like to understand the geometry of Euclidean distance optimization question on $Y \subseteq \mathbb{A}^n$ using its projective closure $X \subseteq \mathbb{P}^n$. We start with smooth projective variety $X \subseteq \mathbb{P}^n$ and consider its affine chart $Y = X \cap \mathbb{A}^n$. Since the geometry of X affects Y, we will make two transversal intersection assumptions on X.

Firstly, for the rest of the chapter, assume X intersects H_∞ transversely. Since H_∞ is a hyperplane, this condition implies that $X \not\subseteq H_\infty$. If we had not assumed this, then we would have $X \subseteq H_\infty$ hence its affine chart Y would be an empty set.

Secondly, for the rest of the chapter, we will assume X intersects $Q_\infty := Q \cap H_\infty$ transversely. The importance of this assumption is in the content of the [17, Lemma 6.7, Lemma 6.8, Proposition 6.10] and it allows us to write $\text{EDdegree}(Y) = \text{EDdegree}(X)$. Essentially, if X would intersects Q_∞ non-transversely, then some critical points of Euclidean distance problem (32) would lie in $X \cap H_\infty$ hence the number would be $\text{EDdegree}(Y) \neq \text{EDdegree}(X)$.

Lemma 5.21. *[17, Lemma 6.7, Lemma 6.8 and Proposition 6.10] Under the assumptions of X intersects H_∞ transversely and X intersects Q_∞ transversely, for a general data point $u \in \mathbb{A}^n$, no complex critical points of the function d_u on Y lie on the hyperplane at infinity H_∞.*

Remark 5.6. Choose $X \subseteq \mathbb{P}^n$ given by $\mathbb{V}(F_1, \dots, F_s)$ and $H_\infty = \mathbb{V}(x_0)$ and let $Y = X \cap \mathbb{A}^n$ where $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$. Recall that in Example 3.12 we have shown that if we choose the quadric $q(x_0, \dots, x_n) = x_0^2 + \sum_{i=1}^n (x_i - u_i x_0)^2$ then affine reciprocal polar variety $W^\perp_{H_\infty}(Y)$ is exactly the critical ideal $C_{Y,u}$ for u generic enough by the 3.10. Therefore we have the observation

$$\text{EDdegree}(Y) = \deg C_{Y,u} = \deg W^\perp_{H_\infty}(Y).$$

Remark 5.7. We furthermore have a theorem connecting the class of reciprocal polar varieties with the classical polar varieties [29, Corollary 10] which states that

$$\left[W_i^{\perp}(X)\right] = \sum_{j=0}^{i} h^{i-j} \cap p_j(X),$$

which implies

$$\deg\left[W_i^{\perp}(X)\right] = \sum_{j=0}^{i} \mu_j(X).$$

Since H_∞ is the last element of the flag, we are going to have $\deg W_m^{\perp}(X) = \sum_{j=0}^{m} \mu_j(X)$.

Theorem 5.22. *[10, Theorem 2.13] Let X be a smooth projective variety in $\mathbb{P}^n$ and $Y = X \cap \mathbb{A}^n$ where $\mathbb{A}^n = \mathbb{P}^n \setminus H_\infty$. Assume the intersections $X_\infty = X \cap H_\infty$ and $X_\infty \cap Q_\infty$ are both transversal then Euclidean distance degree of Y is the sum of polar degrees of X*

$$\text{EDdegree}(Y) = \mu_0(X) + \cdots + \mu_m(X).$$

Proof. First of all, the definition of EDdegree(Y) is precisely defined as the degree of the ideal $C_{Y,u}$ where u is a general data point in $\mathbb{A}^n \setminus Y$. Furthermore using remark 5.6 we have

$$\text{EDdegree}(Y) = \deg W_{H_\infty}^{\perp}(Y).$$

Since the affine reciprocal polar variety is defined to be $W_{H_\infty}^{\perp}(Y) := W_{H_\infty}^{\perp}(X) \cap \mathbb{A}^n$ by definition its degree is exactly $\deg W_{H_\infty}^{\perp}(X)$. Furthermore by Remark 5.7 we have $\deg W_{H_\infty}^{\perp}(X) = \sum_{i=0}^{m} \mu_i(X)$. for generic Q. Since data point u is generic $Q = \mathbb{V}(q)$ defined to be $q(x_0, \ldots, x_n) = x_0^2 + \sum_{i=0}^{n} (x_i - u_i x_0)^2$ is also generic, hence

$$\text{EDdegree}(Y) = \deg W_{H_\infty}^{\perp}(Y) = \deg W_{H_\infty}^{\perp}(X) = \sum_{i=0}^{m} \mu_i(X),$$

proving that Euclidean distance degree of Y is the sum of polar degrees of its projective closure. ■

By the assumption on Lemma 5.21 we have EDdegree(X) = EDdegree(Y), therefore connecting it with the Theorem 5.22, we have the following corollary.

Corollary 5.23. EDdegree$(X) = \sum_{i=0}^{m} \mu_i(X)$.

Example 5.9 (Hypersurface)**.** Let $X \subseteq \mathbb{P}^n$ be smooth hypersurface of degree d, then we have shown that its polar degrees are $\mu_i(X) = d(d-1)^i$ in Example 2.5. Therefore

$$\text{EDdegree}(X) = d\sum_{i=0}^{m}(d-1)^i,$$

by the Theorem 5.22. △

Since we established the properties of polar degrees in Corollary 5.19, by using $\mu_i(X) = \mu_{\alpha(X)-i}(X^\vee)$, we get the following corollary.

Corollary 5.24. EDdegree$(X) =$ EDdegree$(X^\vee)$

Furthermore since we have already shown the relationship between polar degrees and the Chern classes of X in Corollary 5.20, we have the following corollary:

Corollary 5.25. *[17, Theorem 5.8] Let X be a smooth irreducible subvariety of dimension m in $\mathbb{P}^n$, and suppose that X is transversal to the isotropic quadric Q. Then*

$$\mathrm{EDdegree}(X) = \sum_{j=0}^{m} (-1)^j \cdot \left(2^{m-j+1} - 1\right) \cdot \deg\left(c_j(X)\right).$$

Proof. Using Corollary 5.20 we have the following formula of polar degrees

$$\mu_i(X) = \sum_{j=0}^{i} (-1)^j \binom{m-j+1}{m-i+1} \deg\left(c_j(X)\right).$$

By Corollary 5.23 we know that

$$\mathrm{EDdegree}(X) = \sum_{i=0}^{m} \mu_i(X).$$

If we combine these two formulas together we get

$$\mathrm{EDdegree}(X) = \sum_{i=0}^{m} \mu_i(X) = \sum_{i=0}^{m} \left(\sum_{j=0}^{i} (-1)^j \binom{m-j+1}{m-i+1} \deg\left(c_j(X)\right) \right).$$

Now we will make a combinatorial argument on double summation $\sum_{i=0}^{m} \sum_{j=0}^{i}$. Note that it sums over triangle grid of length m. If we think the j as the x axis and i as the y axis in the positive orthant, the summation is sweeping through the rows of a triangle. As long as we trace through the same grid, we will get the same summation. Therefore we can take the summation as $\sum_{j=0}^{m} \sum_{i=j}^{m}$

instead and that would be the summation that sweeps the columns instead. Therefore

$$\begin{aligned}
\mathrm{EDdegree}(X) &= \sum_{i=0}^{m}\left(\sum_{j=0}^{i}(-1)^j\binom{m-j+1}{m-i+1}\deg\left(c_j(X)\right)\right)\\
&= \sum_{j=0}^{m}\left(\sum_{i=j}^{m}(-1)^j\binom{m-j+1}{m-i+1}\deg\left(c_j(X)\right)\right)\\
&= \sum_{j=0}^{m}(-1)^j\deg\left(c_j(X)\right)\left(\sum_{i=j}^{m}\binom{m-j+1}{m-i+1}\right),
\end{aligned}$$

where we regrouped the summations in terms of indexing. Note that

$$\begin{aligned}
\sum_{i=j}^{m}\binom{m-j+1}{m-i+1} &= \binom{m-j+1}{m-j+1}+\binom{m-j+1}{m-j}+\cdots+\binom{m-j+1}{1}\\
&= \sum_{k=1}^{m-j+1}\binom{m-j+1}{k}\\
&= \left(\sum_{k=0}^{m-j+1}\binom{m-j+1}{k}\right)-1\\
&= 2^{m-j+1}-1,
\end{aligned}$$

where in the last equality we used the identity $\sum_{i=0}^{N}\binom{N}{i}=2^N$ from the binomial theorem where $N=m-j+1$. Therefore we get

$$\mathrm{EDdegree}(X)=\sum_{j=0}^{m}(-1)^j(2^{m-j+1}-1)\deg\left(c_j(X)\right),$$

proving the result. ■

5.5 Polyhedral Norm Distance Degree

We will be using [10, Chapter 5] to discuss the distance optimization problem with respect to polyhedral norm. Our main goal for this chapter is to introduce the polyhedral norm optimization, explaining the main idea of the algorithm to solve it and finally prove that its complexity is governed by multidegree using the characterization we have proved in Theorem 5.17. Afterwards in Section 6, we are going to introduce polyhedral norms more carefully.

First note that the Euclidean distance optimization problem given in (32) can be restated such that

$$\text{minimize}\|\mathbf{x}-\mathbf{u}\| \text{ subject to } \mathbf{x}\in Y. \tag{34}$$

In (32) we have used the Euclidean norm $\|\cdot\|$, however we are not restricted to using only Euclidean norm. Let $B = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$ be the unit ball, our main interest is to use a polyhedral norm which is a norm whose unit ball B is centrally symmetric polytope. Conversely for every centrally symmetric polytope P containing the origin in $\mathbb{R}^n$ there is a norm $\|\cdot\|$ such that its unit ball $B = P$ [18, Lemma 2.10]. So after fixing centrally symmetric polytope B containing the origin in $\mathbb{R}^n$ we will be working with every polyhedral norms.

Example 5.10 (Norm ℓ_∞)**.** Denote the ℓ_∞-norm as $\|\cdot\|_\infty$, recall that for $\mathbf{x} \in \mathbb{R}^n$, $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$, where $|\cdot|$ denotes the absolute value. For $\mathbb{R}^2$ the unit ball B using ℓ_∞ norm is a square with indices $b_1 = (1,1), b_2 = (1,-1), b_3 = (-1,1), b_4 = (-1,-1)$. Therefore the polytope $B = \text{conv}\{b_1, b_2, b_3, b_4\}$ where conv denotes the convex hull. Hence the unit ball B is a square, which is centrally symmetric polytope. $\triangle$

After fixing the unit ball B for the polyhedral norm, we can restate the optimization problem (34) as

$$\text{minimize } \lambda \text{ subject to } \lambda \geq 0 \text{ and } (\mathbf{u} + \lambda B) \cap Y \neq \emptyset. \tag{35}$$

Summation with $\mathbf{u}$ is the Minkowski sum, therefore it is the translation of λB. Scaling B with λ is the dilation defined by $\lambda B = \{\lambda b \in \mathbb{R}^n \mid b \in B\}$. Therefore, informally $(\mathbf{u} + \lambda B)$ is a polytope centered around $\mathbf{u}$, where scaling λ controls the size of B. The minimization in the (35) is needed to find the smallest scalar λ, such that it intersects the affine variety Y.

Example 5.11 (Norm ℓ_∞ Optimization)**.** Let $Y = \mathbb{V}(x_1^2 + x_2^2 - 1)$ and $\mathbf{u}_1 = (3,0)$, $\mathbf{u}_2 = (3,3)$. We will do the optimization (34) visualizing with (35). First of all, since B is just a square, $\mathbf{u}_1 + B$ is the translation seen in the Figure 32.

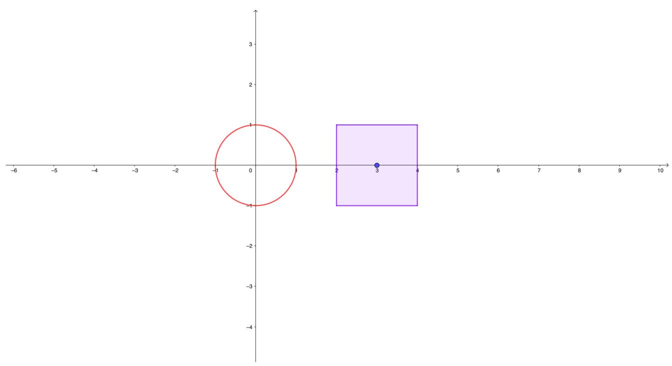

Figure 32: Blue square $\mathbf{u}_1 + B$ and red circle Y

The optimization 35 for the data point $\mathbf{u}_1$ will be the minimum scalar λ such that when the blue square $\mathbf{u}_1 + B$ intersects the red circle Y in the Figure 32. The black point in the Figure 32 is the point that optimized.

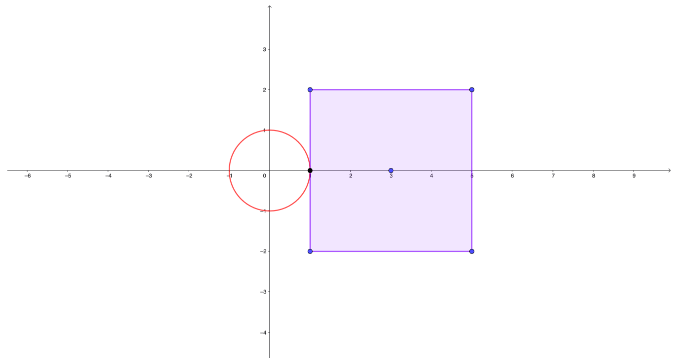

Figure 33: Smallest λ such that $(\mathbf{u}_1 + \lambda B) \cap Y \neq \emptyset$

Similarly for the data point $\mathbf{u}_2$, we can do the exactly the same as shown in the figure below.

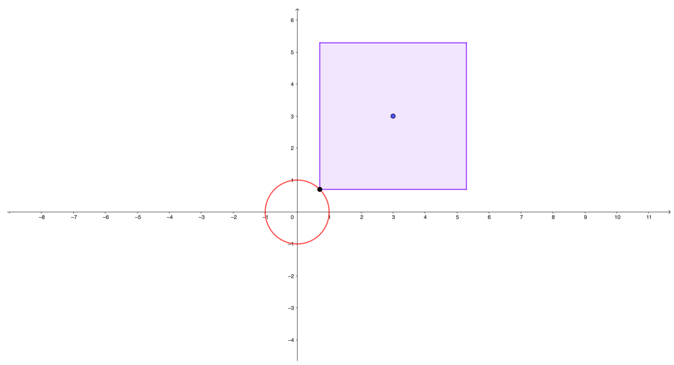

Figure 34: Smallest λ such that $(\mathbf{u}_2 + \lambda B) \cap Y \neq \emptyset$

The key difference between $\mathbf{u}_1$ and $\mathbf{u}_2$ is the which face of the polytope B touching the variety Y. △

Using the notation introduced in [10], we will write L_F for the linear span of the face F of the polytope B in $\mathbb{R}^n$.

Example 5.12. Let $B = \mathrm{conv}\{b_1, b_2, b_3, b_4\}$ be the polytope square in $\mathbb{R}^2$ defined in previous Example 5.10 for the norm ℓ_∞. Given a vertex $F = b_i$, we write $L_F = \mathrm{span}\{b_i\}$. Given an edge $F = \mathrm{conv}\{b_i, b_j\}$ of the polytope B, the $L_F = \mathrm{span}\{b_i, b_j\} = \mathbb{R}^2$.

More generally, for any centrally symmetric polytope B, if the face F has dimension $j < n$, then L_F has dimension $j + 1$, because the origin lies in the interior of B and the dimension of convex hull is defined via the dimension of its affine hull. For instance, if F is a vertex, then L_F is the line through F and the origin. If F is a facet, then $L_F = \mathbb{R}^n$ △

We will now introduce the main idea of the algorithm given in [10] to solve symbolically the optimization problem (34) and connect its complexity with the multidegree, using the characterization of it with polar degrees established in Theorem 5.17. For interested reader on full description of the algorithm, we refer to [10, Section 5.1]. The key lemma that motivates the algorithm is the following.

Lemma 5.26. *[10, Lemma 5.4] Suppose that Y is in general position. Given $\mathbf{u} \in \mathbb{R}^n$, let $\mathbf{x}^*$ be an optimal solution of (34) and let λ^* be its optimal value. Then $\mathbf{x}^*$ is unique, and the point $\frac{1}{\lambda^*}(\mathbf{x}^* - \mathbf{u})$ lies in the relative interior of a unique face F of the polytope B. Let ℓ_F be a linear functional whose maximum over B is attained on F. Then, the optimal point $\mathbf{x}^*$ can be recovered as the unique solution of the optimization problem*

$$\textit{minimize } \ell_F(\mathbf{x}) \textit{ subject to } \mathbf{x} \in (\mathbf{u} + L_F) \cap Y \tag{36}$$

Using the lemma the main idea of the algorithm is the following: Using the discreteness of the unit ball B, we solve the optimization problem (36) for each face F of the polytope B. Among all candidates, we will check which one is the closest to the data point $\mathbf{u}$ by checking whether the point $\frac{1}{\lambda^*}(\mathbf{x}^* - \mathbf{u})$ lies in the relative interior of a unique face F of the polytope B as given in the Lemma above.

Remark 5.8. See that the complexity of the algorithm is governed by the number of faces of the polytope B and the number of complex critical points of the optimization problem (36).

Our main theorem for this chapter is to prove [10, Theorem 5.5] using the description of polar variety we have developed to show that (36) is related to the multidegree.

Recall that we had a description of polar variety $P(X, V)$ when V was given as a kernel of a matrix M in the Lemma. Furthermore when $V \subseteq H_\infty$ we could also write the description of affine polar variety $P(X, V) \cap \mathbb{A}^n$ which was the content of the Lemma 2.19.

Theorem 5.27. *[10, Theorem 5.5] Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $Y = X \cap \mathbb{A}^n$ be the affine variety. Let $L \subset \mathbb{R}^n$ be a general affine-linear space of codimension j and ℓ be a general linear form. The number of complex critical points of ℓ on $L \cap Y$ is the multidegree $\delta_j(X)$.*

Remark 5.9. The statement we give in Theorem 5.27 has an indexing off by one. However the reason for this is the multidegree we define in (17) start with indexing 0, however the multidegree they define in [10, Equation (2.16)] start at indexing 1.

Proof. We are going to prove the statement by showing number of critical points is equal to the polar degree $\mu_{m-j}(X)$ and use the characterization in Theorem 5.17 to show that this is multidegree $\delta_j(X)$.

Let c denote the codimension and m denote the dimension of Y in $\mathbb{A}^n$ and X in $\mathbb{P}^n$. Say the ideal defining Y is $I_Y = \langle f_1, \ldots, f_s \rangle$. Let L be defined by the ideal $\langle k_1, \ldots, k_j \rangle$ where each $k_i(x) = a^{(i)} \cdot x - c^{(i)}$, with $a^{(i)} \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, and $c^{(i)} \in \mathbb{R}$ are the scalars. Let $\ell(x) = c_1 x_1 + \cdots + c_n x_n$ be the linear functional, denote $C = (c_1, \ldots, c_n) \in \mathbb{R}^n$.

The critical points of ℓ on $Y \cap L$ occur when $\nabla \ell$ is linearly dependent on the gradients defining

$Y \cap L$ which is given by the ideal $\langle f_1, \dots, f_s, k_1, \dots, k_j \rangle$. Therefore the critical ideal I_{crit} is

$$I_{\text{crit}} = \langle f_1, \dots, f_s, k_1, \dots, k_j \rangle + \left\langle (c+j+1) \times (c+j+1) \text{ minors } \begin{pmatrix} \mathcal{J}_Y \\ \mathcal{M}' \end{pmatrix} \right\rangle,$$

where $\mathcal{M}'$ is $(j+1) \times n$ matrix whose rows are $C, a^{(1)}, \dots, a^{(j)}$. The minors are taken from $c+j+1$ since for generic Y and generic L which has codimension c and j their intersection will have codimension $c+j$, and the plus 1 is from the Lagrange Multipliers.

Because the linear form ℓ and the affine-linear space L are chosen to be general, their gradient vectors are linearly independent, hence, the matrix $\mathcal{M}'$ has full row rank $j+1$.

Write the matrix $\mathcal{M} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathcal{M}' \end{pmatrix}$ and note that it satisfies $\mathbb{P}(\ker \mathcal{M}) \subseteq H_\infty = \mathbb{V}(x_0)$, has full row rank $j+2$ hence $\dim \ker \mathcal{M} = (n+1) - (j+2) = n-j-1$. After writing $V := \mathbb{P}(\ker \mathcal{M}) \subseteq H_\infty$ we see that $\dim V = n-j-2$ and $P(X,V)$ is a $(m-j)$-th polar variety.

Since $V := \mathbb{P}(\ker \mathcal{M}) \subseteq H_\infty$, we can use the lemma 2.19 to say that the ideal description of affine polar variety $P(X,V) \cap \mathbb{A}^n$ is

$$\langle f_1, \dots, f_s \rangle + \left\langle (c+j+1) \times (c+j+1) \text{ minors } \begin{pmatrix} \mathcal{J}_Y \\ \mathcal{M}' \end{pmatrix} \right\rangle.$$

Intersecting this affine polar variety with codimension j affine linear subspace we get precisely the critical ideal I_{crit}, in other words the critical ideal defines $(P(X, \mathbb{P}(\ker \mathcal{M})) \cap \mathbb{A}^n) \cap L$. Let $K = \overline{L}$ the projective closure of L. We can rewrite it as

$$(P(X,V) \cap K) \cap \mathbb{A}^n.$$

By Corollary 2.13 we know that $\operatorname{codim}_X P(X,V) = m-j$, we have already defined $\operatorname{codim}_{\mathbb{P}^n}(X) = c$ therefore $\operatorname{codim}_{\mathbb{P}^n} P(X,V) = m+c-j = n-j$. By assumption $\operatorname{codim}_{\mathbb{P}^n} K = j$ hence $P(X,V) \cap K$ are finitely many points and since they are in complementary dimensions this number is exactly $\deg P(X,V) = \mu_{m-j}(X)$. Furthermore as everything in general position we have $\deg P(X,V) = \deg(P(X,V) \cap \mathbb{A}^n)$. Hence, the number of points defined in I_{crit} is exactly $\mu_{m-j}(X)$ which is equal to $\delta_j(X)$ by the Theorem 5.17. ■

Definition 5.11 (f-vector)**.** [10, Definition 5.2] Let $B \subset \mathbb{R}^n$ be a polytope. The f-vector of B is $f(B) = (f_0, f_1, \dots, f_{n-1})$, where f_i denotes the number of i-dimensional faces of B, for $0 \leq i \leq n-1$.

Corollary 5.28. *[10, Corollary 5.6] If the variety Y is in general position, then the total number of critical points of the optimization problem (34) while using the algorithm given in [10, Section*

5.1] arises from the f-vector of B and the multidegree of X. That number equals to

$$\delta_{n-1}(X) \cdot f_0(B) + \delta_{n-2}(X) \cdot f_1(B) + \cdots + \delta_0(X) \cdot f_{n-1}(B). \tag{37}$$

Proof. The total number of critical points is $\alpha_0 \cdot f_0(B) + \cdots + \alpha_{n-1} \cdot f_{n-1}(B)$, where α_i is the number of critical points of (36) for a face of dimension i.

Fix a face F of dimension i, then the affine linear subspace $L = \mathbf{u} + L_F$ is dimension $i+1$ since $\dim L_F = \dim F + 1$ and L is a translation. The codimension of L is $n-1-i$, hence using the Theorem 5.27 we have $\alpha_i = \delta_{n-1-i}$. Therefore the total number of critical points is exactly (37). ■

6 Future Directions

In this section, we follow introduction given in [18] closely, and the main goal is to define higher-order distance loci for polyhedral norms and showcase computations for curves in $\mathbb{A}^3$.

We have already started a discussion of polyhedral norms in Section 5.5. As before, let B be our unit ball, $Y \subseteq \mathbb{R}^n$ be smooth codimension one variety given by $\mathbb{V}(f)$. In Section 5.5, we assumed that B was a centrally symmetric, full dimensional polytope in $\mathbb{R}^n$. The polytope B defines a norm $\|.\|_B$, in the following way.

Let $\mathcal{L} \subseteq (\mathbb{R}^n)^*$ be a finite set of linear functionals. Define the polyhedron B as

$$B = \{x \in \mathbb{R}^n \mid \forall \ell \in \mathcal{L}, \ell(x) \leq 1\}.$$

Note that this is a polyhedron given in half-space representation, which contains the origin as well. For the rest of the chapter, we are going to assume $\mathcal{L} = -\mathcal{L}$ so that for polyhedron also implies $B = -B$ which means it is centrally symmetric. We further assume $\mathcal{L}$ also has linear functionals such that B is bounded and full dimensional, which implies B is full dimensional polytope. Then we define polyhedral norm as $h_{\mathcal{L}}(x) = \max_{\ell \in \mathcal{L}} \ell(x)$.

Example 6.1 (Norm ℓ_∞)**.** We recalled in Example 5.10 that $\|x\|_\infty = max\{|x_1|, \ldots, |x_n|\}$. Therefore $\mathcal{L}$ for the ℓ_∞ norm is $\mathcal{L} = \{\pm x_1, \ldots, \pm x_n\}$. Furthermore, its unit ball is the n-cube $[-1,1]^n$. △

Example 6.2 (Norm ℓ_1)**.** The ℓ_1 norm defined to be $\|x\|_1 = \sum_{i=1}^n |x_i|$. Its unit ball is called the cross polytope, and for $n = 3$ the set of linear functionals $\mathcal{L}$ for ℓ_1 norm is given by $\mathcal{L} = \{\pm(x+y+z), \pm(x-y+z), \pm(x+y-z), \pm(x-y-z)\}$. △

Given a polytope B, we denote the set of proper faces as $\mathcal{F}_B$, these are the faces such that $\emptyset, P \notin \mathcal{F}_B$. A hyperplane $H \subseteq \mathbb{R}^n$ is a supporting hyperplane of a polytope P at F if $H \cap B = F$ for some $F \in \mathcal{F}$ and $P \subset H^+$ where $H^+ := \{x \in \mathbb{R}^n \mid H(x) \geq 0\}$.

Definition 6.1 (Inner normal cone)**.** Given a full dimensional polytope B, we define $C(B,F)$ as the cone of all the inner normal vectors of the supporting hyperplanes H of B such that $B \cap H = F$. For $p \in \mathbb{R}^n$, we denote $C_p(B,F) := p + C(B,F)$ for the translation of it.

For a polytope B with face F we denote the B^* for the dual polytope and F^* for the corresponding dual face. For more details on polyhedras and polytopes, we refer reader to the textbook on discrete geometry such as [39].

For the rest of this section, we assume Y is a smooth manifold embedded in $\mathbb{R}^n$ of dimension m. Let $u \in \mathbb{R}^n \setminus Y$ be a data point. With these notations, we can write the distance optimization problem for polyhedral norms as

$$\text{minimize } h_{\mathcal{L}}(u-p) \text{ subject to } p \in Y. \tag{38}$$

Denote $d(u,Y) = \min_{p \in Y} h_{\mathcal{L}}(u-p)$, note that this is the same question as the optimization question, we defined in (35) in the Section 5.5.

Now we are going to ask the inverse problem of distance optimization question. For $p \in Y$, what are all the data points $u \in \mathbb{R}^n$ such that the optimization problem (38) is minimized at the point p. This motivates the definition of Voronoi cells:

Definition 6.2. [18, Definition 3.1] The Voronoi cell of $p \in Y$ with respect to polyhedral norm $h_{\mathcal{L}}$ is

$$\mathrm{Vor}_Y(p) := \{u \in \mathbb{R}^n \mid d(u,Y) = h_{\mathcal{L}}(u-p)\}.$$

This is the set of points in $\mathbb{R}^n$, whose closest point on the manifold Y with respect to the polyhedral norm is p.

By Lemma 5.26, given $u \in \mathbb{R}^n \setminus Y$, the optimization problem (38), is solved at the point $y^* \in Y$ for a unique face F. This motivates us to record the face where (38) is optimized at.

Definition 6.3. [18, Definition 3.2] Let Y be a variety and let N_pY denote the normal space of Y at point p placed at the origin. Define the type of $p \in Y$ as:

$$\mathrm{Type}(p) := \{F \in \mathcal{F}_B \mid C(B,F) \cap N_pY \neq \emptyset\}.$$

Proposition 6.1. *[18, Proposition 3.4] For a smooth real variety Y and $p \in Y$, the Voronoi cell satisfies*

$$\mathrm{Vor}_Y(p) \subset \bigcup_{F \in \mathrm{Type}(p)} C_p(B^*, F^*).$$

Remark 6.1. Although we are working with a polyhedral norm, the geometry of the Voronoi cell and polyhedral distance optimization in particular, depends on the Euclidean normal space of the manifold Y. This is because the definition of $\mathrm{Type}(p)$ requires the normal space N_pY.

Remark 6.2. The Euclidean distance counterpart of the Proposition 6.1 is the [10, Proposition 8.2], where it states that Voronoi cell for Euclidean distance is contained in the set $p + N_pY$ which is the normal space centered on p.

See that given $u \in \mathbb{R}^n \setminus Y$, the necessary condition of u being optimized with respect to polyhedral distance at the point $p \in Y$ is $u \in \bigcup_{F \in \mathrm{Type}(p)} C_p(B^*, F^*)$. This motivates us to define the Voronoi cones.

Definition 6.4 (Voronoi Cone)**.** Given a smooth manifold $Y \subseteq \mathbb{R}^n$ and a polyhedral norm with unit ball B, we define Voronoi cone at the point $p \in Y$ as

$$\mathrm{VCone}(p) := \bigcup_{F \in \mathrm{Type}(p)} C_p(B^*, F^*)$$

6.1 Higher Order Polyhedral Norm Loci

The idea of this subsection is to write the higher order analogs of the construction in previous section. As everything is in the affine setting, we need to replace the definitions given in Section 4 by their affine versions. The content for the affine higher order normal space was discussed in [31, 8. Higher-order distance degrees of affine morphisms].

Consider an algebraic morphism $\varphi : \mathbb{A}^m \to \mathbb{A}^n$, therefore there exists $\sigma_1, \ldots, \sigma_n \in \mathbb{C}[t_1, \ldots, t_m]$ such that $\varphi = (\sigma_1, \ldots, \sigma_n)$. Note that in Section 4, we started discussion for embeddings of the form $X \hookrightarrow \mathbb{P}^n$, where X was a projective variety and built the theory of osculating spaces locally. In this section, we work solely in affine space $\mathbb{A}^n$, and our main variety is will be $Y := \overline{\varphi(\mathbb{A}^m)} \subseteq \mathbb{A}^n$.

Similar to construction in Definition 4.5, we will write the k-th jet matrix as

$$A_p^{(k)}(\varphi) := \left(\frac{1}{|\alpha|} \frac{\partial^{|\alpha|} \varphi_i}{\partial t^\alpha}(p) \right)_{\substack{\alpha \in \mathbb{N}^m \\ 1 \le |\alpha| \le k \\ 1 \le i \le n}}, \quad p \in \mathbb{A}^m.$$

Note that the difference from the projective version is $|\alpha|$ does not start from 0. In other words, the size of the matrix $A_p^{(k)}(\varphi)$ is $(\binom{m+k}{k} - 1) \times n$. Therefore, the first row of $A_p^{(k)}(\varphi)$ is not the vector obtained by evaluating φ at point p. Similar to projective osculating spaces, the affine analog is the following:

Definition 6.5. [31, Definition 8.1] For $p \in \mathbb{A}^m$, the k-th order affine osculating space of φ at $\varphi(p) \in \varphi(\mathbb{A}^m)$ is

$$T_p^k(\varphi) = \mathrm{row}\, A_p^{(k)}(\varphi).$$

For generic $p \in \mathbb{A}^m$ the dimension of $T_p^k(\varphi)$ is constant. Define the affine generic k-th order osculating dimension to be $m_k = \mathrm{rk}\, A_p^{(k)}(\varphi)$ for generic $p \in \mathbb{A}^m$. Let U_k be the dense open subset of the points $p \in \mathbb{A}^m$ such that m_k is constant for generic $p \in \mathbb{A}^m$. Therefore in the set U_k, the dimension of the k-th order osculating space is $m_k = \dim T_p^k(\varphi)$. We say that morphism φ is globally k-osculating if $U_k = \mathbb{A}^m$ and k-regular if it is k osculating and $m_k = \binom{m+k}{k} - 1$.

Remark 6.3. Note that U_k is the affine version of X_k° in the Definition 4.8. The affine osculating space $T_p^k(\varphi)$ always passes through the origin. The affine generic k-osculating dimension m_k differs by one from the projective generic k-osculating dimension defined via X_k° also denoted by m_k.

Example 6.3. Let $Y \subseteq \mathbb{A}^n$ be a curve given by $Y = \overline{\varphi(\mathbb{A}^1)}$. Assume that φ is k-regular, then the dimension of the k-th osculating space is $m_k = \binom{m+k}{k} - 1 = \binom{1+k}{k} - 1 = k$. Therefore the dimension of the k-th osculating space is k. △

In order to define the normal space at a point p, we pass to the compactification $\mathbb{P}^n = \mathbb{A}^n \cup H_\infty$ and do the same construction of normal space in Definition 3.4. That is, given linear subspace $L \subseteq \mathbb{A}^n$ and a point $p \in \mathbb{A}^n$, the orthogonal complement of L passing through the p is

$$p + L^\perp := (p + L_\infty^\perp) \cap \mathbb{A}^n,$$

where $L_\infty = \mathbb{P}(L) \cap H_\infty$. This is exactly what we have done in the Example 3.8.

Definition 6.6. [31, Definition 8.2] For $p \in U_k$, the k-th order normal space of (φ, Q) at $\varphi(p)$ is

$$N_p^k(\varphi, Q) := \left(T_p^k(\varphi)\right)^\perp \subseteq \mathbb{A}^n.$$

For $k = 1$, this is called the normal space of (φ, Q) at p and denoted as $N_p(\varphi, Q)$.

We always work with $Q = Q_{ED}$, therefore the normal space $N_p^k(\varphi, Q)$ is the orthogonal complement of the osculating space $T_p^k(\varphi)$. Therefore, we will drop the notation Q from the definition and write $N_p^k(\varphi)$ for the normal space at the point p passing through the origin.

Example 6.4. Continuing the Example 6.3, we have found the $\dim T_p^k(\varphi) = m_k = k$ for φ defining a curve in $\mathbb{R}^n$. Therefore the dimension of the normal space is $\dim N_p^k(\varphi) = n - m_k = n - k$. Therefore, for a curve given by k-regular map $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$, where $k = n - 1$ always define one dimensional normal space. △

The part where normal space was used in the polyhedral optimization question was in Type and VCone. Therefore, we will define the higher order versions of it as follows.

Definition 6.7. Let $\varphi : \mathbb{A}^m \to \mathbb{A}^n$ be a morphism. Define the k-th order type of $p \in \overline{\varphi(\mathbb{A}^m)}$ as

$$\mathrm{Type}^{(k)}(p) = \{F \in \mathcal{F}(B) \mid N_p^k(\varphi) \cap C(B, F) \neq \emptyset\}.$$

Definition 6.8. Let $\varphi : \mathbb{A}^m \to \mathbb{A}^n$ be a morphism. Define the k-th order Voronoi cone at the point $p \in \overline{\varphi(\mathbb{A}^m)}$ as

$$\mathrm{VCone}^{(k)}(p) = \bigcup_{F \in \mathrm{Type}^{(k)}(p)} C_p(B^*, F^*).$$

Similar to definition of higher order distance loci with respect to quadric Q given in [31, Definition 4.6], we are going to define polyhedral analogue of it.

Definition 6.9 (Higher order Polyhedral distance loci)**.** Let $\varphi : \mathbb{A}^m \to \mathbb{A}^n$ be a morphism where $Y = \overline{\varphi(\mathbb{A}^m)}$. Then we define the k-th order polyhedral distance loci is the set

$$\mathcal{PL}_k(\varphi) = \bigcup_{p \in Y} \mathrm{VCone}^{(k)}(p).$$

Remark 6.4. Note that in [31, Definition 4.6] the higher order distance loci for Euclidean distance was given by

$$\overline{\bigcup_{p \in X_k^\circ} \mathbb{N}_p^k(\varphi, Q)}.$$

Since the Voronoi diagram for Euclidean distance was contained in the normal space, this motivated us to define the higher order polyhedral distance loci using $\mathrm{VCone}^{(k)}(p)$.

Example 6.5. Consider the real smooth curve X given by the image of the map $t \mapsto (t, t^2, t^3)$. Let the polyhedral norm be ℓ, hence the unit ball is the three dimensional cube $B = C_3$.

Recall that taking the dual of a polytope is an operation that interchanges the facets and vertices. Therefore, the dual of the cube B is the cross polytope B^*. Since $(\pm 1, \pm 1, \pm 1)$ are the 8 vertices of the cube B the normal vectors of the cross polytope will be given by these 8 linear functionals as in the Example 6.2.

We compute the second order polyhedral distance loci $\mathcal{PL}_2$ in this example, using the geometry of the cube B and its dual B^* crosspolytope. Recall that in Example 4.14, we have showed the general formula of the osculating plane for a general degree 3 curve in $\mathbb{R}^3$. Since the twisted cubic curve is given by the coefficient vectors

$$\vec{a} = (0, 0, 0), \vec{b} = (1, 0, 0), \vec{c} = (0, 1, 0), \vec{d} = (0, 0, 1),$$

we obtain the equation for the second order osculating space passing through the origin at a fixed t to be

$$(12t^2, -12t, 4) \cdot (x, y, z) = 0$$

where $\cdot$ denotes the dot product. The normal direction of the second order osculating plane is the generator of second order normal space $N_p^2(f)$, since we are using quadric Q_{ED}. Therefore the direction of the normal space for a fixed t is

$$(12t^2, -12t, 4).$$

We subdivide the parameter space $\mathbb{A}^1 = \mathbb{R}$ into intervals I_i such that for each $t \in I_i$, the line from the origin $(0, 0, 0)$ to $(12t^2, -12t, 4)$ intersects with the i-th face of the cross polytope B^*.

Since B is centrally symmetric, B^* is also centrally symmetric. Therefore, if the lin intersects i-th face, it also intersects the $-i$-th face. We use the geometry of cross polytope to subdivide $\mathbb{R}$.

See that since for a fixed t the line can be associated to a sign vector. That is, we take the sign function component wise. Considering the vector $(12t^2, -12t, 4)$, the first and the last components are positive for any $t \in \mathbb{R}$. However for $t > 0$ the sign vector is $(+, -, +)$ and for the $t < 0$ the sign vector is $(+, +, +)$. Lastly in the case of $t = 0$ we have the sign vector $(0, 0, +)$. Therefore we have subdivided the parameter space $\mathbb{R}$ into

$$I_1 = (-\infty, 0),\ I_2 = \{0\},\ I_3 = (0, \infty).$$

For $t \in I_1$ the only facet of the cross polytope it touches is the direction $(+, +, +)$ and $(-, -, -)$, this means for $t \in I_1$ we are going to attach the line in the direction of $(1, 1, 1)$. Similarly for $t \in I_3$ the only facet of the cross polytope it touches is the direction $(+, -, +)$ and $(-, +, -)$, this means for $t \in I_3$, we are going to attach the lines in the direction of $(1, -1, 1)$. Lastly for $t \in I_2 = \{0\}$, the line touches the vertex $(0, 0, 1)$ and $(0, 0, -1)$ of the cross polytope. Note that, vertex $(0, 0, \pm 1)$ of the cross polytope has 4 facets supporting it, therefore the inner normal cone will be the cone

$$C = \text{cone}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

For the vertex $(0, 0, -1)$, we also have the cone $-C$. The figure we get without the cone C and $-C$ is:

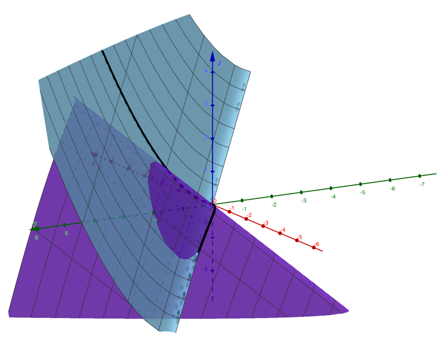

Figure 35: Distance Loci without C and $-C$

The figure obtained, when C and $-C$ included is the following:

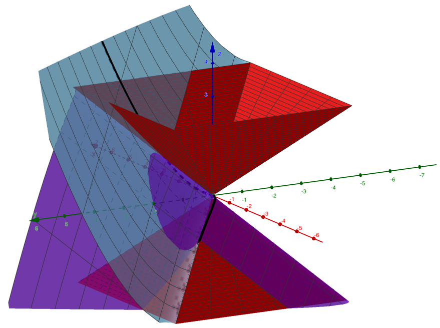

Figure 36: Distance Loci with C and $-C$

In the figures above, the turquoise surface is generated by a pencil of lines with the parametric equation

$$(u, v) \mapsto (u, u^2, u^3) + (1, -1, 1)v,$$

where $u > 0$ and $v \in \mathbb{R}$.

Similarly the purple surface in the figures above is generated by a pencil of lines with the parametric equation

$$(u, v) \mapsto (u, u^2, u^3) + (1, 1, 1)v,$$

where $u < 0$ and $v \in \mathbb{R}$.

Furthermore, the reason looking at the signs worked in this example is that the dual of the cube is the cross-polytope. Specifically, for the cross-polytope, each facet is a simplex that corresponds to a specific sign vector.

Lastly, we note that the intersection of these surfaces yields the higher order analogue of the medial axis for the polyhedral optimization problem. $\triangle$

Remark 6.5. One possibility is to refine the definition of $\mathcal{PL}_k(\varphi)$, where indexing is not all $p \in Y$, but varies accordingly with the dimension of $\mathrm{VCone}^{(k)}(p)$. Note that this leads to a discussion of the stratification of the Y into parts where the dimension of the Voronoi cone changes. For details of codimension 1 manifolds with first order normal space we refer to [18, 4 Stratification of Codimension-One Manifolds]. Here, although we are working with a curve rather than codimension-one manifold, its $n-1$th higher order normal space is also dimension 1 as discussed in the Example 6.4. Therefore a similar approach can be applied.

Now we are going to introduce an ideal J_F defined in [18, 5 Medial Axis]. Although, we do are not going to get into the discussion of the medial axis, the ideal J_F will allow us to find a closed subset with respect to Zariski topology that contains $\mathcal{PL}_k(\varphi)$.

First, we define the ideal for $k = 1$ as given in [18]. Assume Y is a smooth codimension 1 affine variety $Y = \mathbb{V}(f)$. Let $F \in \mathcal{F}_B(i)$ be a proper face of the unit ball B of codimension i. Then there is $\{\ell_1, \dots, \ell_i\} \subseteq \mathcal{L}$ such that they define the face F. Consider the ideal

$$J_F = \langle f(x) \rangle + \langle \ell_1(x-u) - \lambda, \dots, \ell_i(x-u) - \lambda \rangle + \left\langle (i+1) \times (i+1) \text{ minors of } \begin{pmatrix} \nabla f \\ \nabla \ell_1 \\ \vdots \\ \nabla l_i \end{pmatrix} \right\rangle$$

of $\mathbb{R}[x_1, \dots, x_n, u_1, \dots, u_n, \lambda]$. Note that if F is a vertex, which has codimension $i = n$, then the matrix in the minor condition would be $(n+1) \times n$ a matrix; hence, there would be no $(n+1) \times (n+1)$ minors. So J_F in this case it would not contain any determinantal equations.

Write $V_F = \operatorname{span}\{\nabla \ell_1, \dots, \nabla \ell_i\}$. The minor condition in the ideal J_F is the condition that $N_p Y \subseteq V_F$ equivalently $N_p Y \cap V_F \neq \emptyset$ nontrivially. We note that the ideal J_F defines the geometric set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ where $p \in Y$, $N_p Y \cap V_F \neq \emptyset$, and the value $\lambda \in \mathbb{R}$ gets the same value for each supporting hyperplane of F. Note that,

$$C(B, F) = \operatorname{cone}\{\nabla \ell_1, \dots \nabla \ell_i\} \subseteq \operatorname{span}\{\nabla \ell_1, \dots \nabla \ell_i\} = V_F,$$

therefore, the condition $F \in \operatorname{Type}(p)$ which is $N_p Y \cap C(B, F) \neq \emptyset$ implies $N_p Y \cap V_F \neq \emptyset$. Hence J_F is an ideal containing all correspondence between p, u, and λ such that polyhedral optimization is realized at the face F with value λ.

Similarly, we are going to define $J_F^{(k)}$ that records higher order correspondence. We restrict ourselves to curves $Y = \overline{\varphi(\mathbb{A}^1)}$ given by morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$ that are $(n-1)$-regular. As discussed in the Example 6.4, we get the dimension of the second order normal space to be 1. Write $\eta_{n-1}(t)$ for the vector that depends on t, that gives the direction of the $(n-1)$-th order normal space. For example, for φ defining twisted cubic curve, $\eta_2(t) = (12t^2, -12t, 4)$. Define $J_F^{(k)}$ as

$$\langle \ell_1(\varphi(t) - u) - \lambda, \dots, \ell_i(\varphi(t) - u) - \lambda \rangle + \left\langle (i+1) \times (i+1) \text{ minors of } \begin{pmatrix} \eta_{n-1}(t) \\ \nabla \ell_1 \\ \vdots \\ \nabla \ell_i \end{pmatrix} \right\rangle.$$

Now similarly to the J_F this ideal defines a geometric set $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ such that $t \in \mathbb{A}^1$, where $(n-1)$-th order normal space $N_p^{(n-1)}(f)$ intersects nontrivially

$$N_p^{(n-1)}(f) \cap V_F \neq \emptyset,$$

and supporting hyperplanes of face F all get the same value. Similar to the previous case, we have the condition $F \in \operatorname{Type}^{n-1}(p)$ which is $N_p^{(n-1)} Y \cap C(B, F) \neq \emptyset$ implies $N_p^{(k-1)} Y \cap V_F \neq \emptyset$. Define

$J^{(n-1)} = \bigcap_{F \in \mathcal{F}_B} J_F^{(n-1)}$. Now we will eliminate the variables t and λ to define

$$I^{(n-1)} := J^{(n-1)} \cap \mathbb{R}[u_1, \dots, u_n].$$

The real variety $\mathbb{V}(I^{(n-1)})$ is the algebraic closure of the set $u \in \mathbb{R}^n$, such that there is a face $F \in \mathcal{F}_B$, and linear functionals $\{\ell_1, \dots, \ell_i\} \subseteq \mathcal{L}$ defining F we have

$$N_p^{n-1}(f) \cap V_F \neq \emptyset.$$

and they all have the same value λ, at the same point $\varphi(t)$ for some $t \in \mathbb{A}^1$.

Theorem 6.2. *Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$, be $(n-1)$-regular algebraic morphism. Let $\eta_{n-1}(t)$ be the vector that depends on variable t that gives the direction of the $(n-1)$-th order normal space $N_{\varphi(t)}^{(n-1)}(\varphi)$ at $\varphi(t)$. Given a proper face $F \in \mathcal{F}_B$, with $\{\ell_1, \dots, \ell_i\} \subseteq \mathcal{L}$ linear functionals defining F, define the ideal*

$$J_F^{(n-1)} = \langle \ell_1(\varphi(t) - u) - \lambda, \dots, \ell_i(\varphi(t) - u) - \lambda \rangle + \left\langle (i+1) \times (i+1) \text{ minors of } \begin{pmatrix} \eta_{n-1}(t) \\ \nabla \ell_1 \\ \vdots \\ \nabla \ell_i \end{pmatrix} \right\rangle$$

in $\mathbb{R}[t, u_1, \dots, u_n, \lambda]$. Define $J^{(n-1)} := \bigcap_{F \in \mathcal{F}_B} J_F^{(n-1)}$ and define the eliminated ideal $I^{(n-1)} := J^{(n-1)} \cap \mathbb{R}[u_1, \dots, u_n]$. Then we have

$$\mathcal{PL}_{n-1}(\varphi) \subseteq \mathbb{V}(I^{(n-1)}).$$

Proof. First of all, since B is a polytope, there is a vertex representation of it, say $\{v_1, \dots, v_k\} \subseteq B \subseteq \mathbb{R}^n$ is the minimal set of points in B such that B is the convex hull $B = \operatorname{conv}\{v_1, \dots, v_k\}$.

Now by definition $\mathcal{PL}_{n-1}(\varphi) = \bigcup_{t \in \mathbb{A}^1} \operatorname{VCone}(\varphi(t))$. Let $u \in \mathcal{PL}_{n-1}(\varphi)$, then there is $t \in \mathbb{A}^1$ such that $u \in \operatorname{VCone}^{(n-1)}(p)$ where $p = \varphi(t)$. Therefore there exist $F \in \operatorname{Type}^{(n-1)}(p)$, such that $u \in C_p(B^*, F^*)$, which means $u - p \in C(B^*, F^*)$. Let $\{\ell_1, \dots, \ell_i\} \subseteq \mathcal{L}$ be the linear functionals defining the face F, and let $\{v_{j_1}, \dots v_{j_m}\} \subseteq \{v_1, \dots, v_k\}$ be the minimal subset of vertices of B, where the face F is the convex hull of these points $F = \operatorname{conv}\{v_{j_1}, \dots v_{j_m}\}$. Note that by definition any linear functional $\ell \in \{\ell_1, \dots, \ell_i\}$, evaluates to 1 in the vertex set of F, that is $\ell(v) = 1$ where $v \in \{v_{j_1}, \dots v_{j_m}\}$.

Since taking dual of a polytope B changes vertices to facets, and facets to vertices, we have $C(B^*, F^*) = \mathit{cone}\{v_{j_1}, \dots v_{j_m}\}$ therefore $u - p = \lambda_1 v_{j_1} + \dots + \lambda_m v_{j_m}$ for some coefficients $\lambda_1, \dots, \lambda_m \geq 0$. Note that for any linear functional $\ell \in \{\ell_1, \dots, \ell_i\}$ we have

$$\ell(u - p) = \ell(\lambda_1 v_{j_1} + \dots + \lambda_m v_{j_m}) = \lambda_1 \ell(v_{j_1}) + \dots + \lambda_m \ell(v_{j_m}) = \lambda_1 + \dots + \lambda_m.$$

This argument shows $u - p \in C(B^*, F^*)$ implies that there exists a scalar λ, such that for all $j = 1, \dots i$, we have $\ell_j(\varphi(t) - u) = \lambda$. Furthermore, since $F \in \mathrm{Type}^{(n-1)}(p)$ by definition we have $N_p^{(n-1)}(\varphi) \cap C(B, F) \neq \emptyset$ which implies $N_p^{(n-1)}(\varphi) \cap V_F \neq \emptyset$ where $V_F = \mathrm{span}\{\nabla\ell_1, \dots, \nabla\ell_i\}$. This shows that $(t, u, \lambda) \in \mathbb{V}(J_F^{(n-1)}) \subseteq \mathbb{V}(J^{(n-1)})$, therefore after eliminating the variables t and λ we get $u \in \mathbb{V}(I^{(n-1)})$. ■

Example 6.6. We did a calculation of $J_F^{(2)}$ for each proper face $F \in \mathcal{F}_B$ for the twisted cubic curve example. Using Julia programming language [8] with OSCAR package [26, 12] package, we used algorithm below:

```
using Oscar
R, (t, u1, u2, u3, lambda) = polynomial_ring(QQ, [:t,:u1,:u2,:u3,:lambda])
B = cube(3)
phi = [t,t^2,t^3] #twisted cubic curve example
u = [u1,u2,u3] #data point
n2 = matrix(R, 1,3, [12t^2 , -12t , 4]) #second order normal space direction
list_of_ideals = []
A, b = halfspace_matrix_pair(facets(B))
L = [A[i, :] / b[i] for i in 1:length(b)]
for d in 0:(Oscar.dim(B)-1) #iterating over all proper faces of B
    for F in faces(B,d)
        L_F = []
        for l in L #finding the linear functionals for the face
            if all(dot(l, v) == 1 for v in vertices(F))
                push!(L_F, l)
            end
        end
        elem = ideal(R, [(dot(l, phi - u) - lambda) for l in L_F]) #first part
        mat = vcat(n2, matrix(R, [l for l in L_F]))
        deter = ideal(R, minors(mat, length(L_F) + 1)) #determinantal part

        J_F = elem + deter
        I_F = eliminate(J_F, [t,lambda])

        println(vertices(F))
        println(gens(I_F))
        println()
        push!(list_of_ideals, I_F)
    end
end

datalocus = reduce(intersect, list_of_ideals)
println(gens(datalocus))
```

Due to the full dimensional cones centered at the origin as in the Example 6.5, variety $\mathbb{V}(I^{(2)})$

is the whole space $\mathbb{A}^3$. After not using the facets in the iteration (which is just changing "Oscar.dim(B) -1" to "Oscar.dim(B) -2") we get a nontrivial set defined by only one polynomial with variables u_1, u_2 and u_3:

$$
\begin{gathered}
(u_2 - u_3)(u_1^3 + 3u_1^2u_2 + 3u_1u_2^2 - u_1u_2 - u_1u_3 + u_2^3 - 2u_2^2 - 3u_2u_3 - u_3^2) \\
(u_1^3 - 3u_1^2u_2 + 3u_1u_2^2 - u_1u_2 + u_1u_3 - u_2^3 + 2u_2^2 - 3u_2u_3 + u_3^2) \\
(u_1^3 + 3u_1^2u_2 - 2u_1^2 + 3u_1u_2^2 + u_1u_2 - 5u_1u_3 + u_2^3 + 2u_2^2 - 3u_2u_3 + 2u_2 - u_3^2 - 2u_3)(u_1 + u_3) \\
(u_1^3 - 3u_1^2u_2 + 2u_1^2 + 3u_1u_2^2 + u_1u_2 + 5u_1u_3 - u_2^3 - 2u_2^2 - 3u_2u_3 - 2u_2 + u_3^2 - 2u_3)(u_1 - u_3)(u_2 + u_3) = 0.
\end{gathered}
$$

Note that there are 8 factors of this polynomial. If we plot everything together we get the following figure:

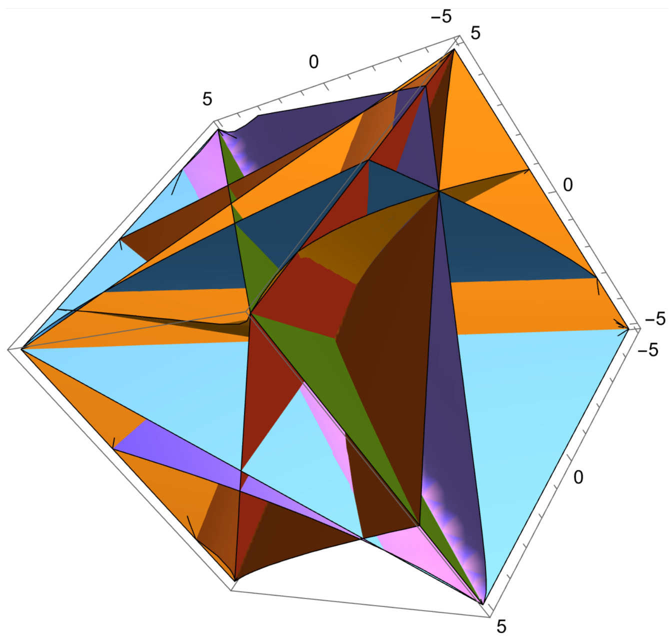

Figure 37: Without the facet computation of $\mathbb{V}(I^{(2)})$

If we plot without the linear components, we get the following figure instead:

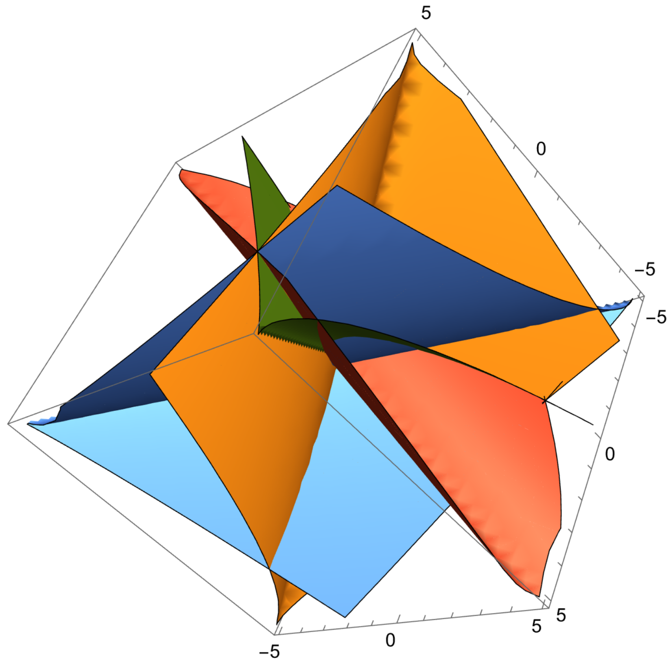

Figure 38: Without the linear components

Note that four of the non-linear factors, define almost identical pieces, and can be thought as ruled surfaces for the twisted cubic curve. $\triangle$

Remark 6.6. Note that in the proof of the Theorem 6.2, we did not not have to use all the proper faces $F \in \mathcal{F}_B$, because, we only worked over faces $F \in \mathrm{Type}^{(n-1)}(\varphi(t))$ for some $t \in \mathbb{A}^1$. This motivates us to refine the Theorem 6.2, by defining $J^{(n-1)}$ not as intersection over all proper faces $\mathcal{F}_B$ but a smaller subset of it $\mathcal{S}_B$ defined as $\mathcal{S}_B := \{F \in \mathcal{F}_B \mid \exists t \in \mathbb{A}^1, F \in \mathrm{Type}^{(n-1)}(\varphi(t))\}$. Then we get the following corollary with the same proof as in the Theorem 6.2.

Corollary 6.3. *Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$, be $(n-1)$-regular algebraic morphism. Let $\eta_{n-1}(t)$ be the vector that depends on variable t that gives the direction of the $(n-1)$-th order normal space $N^{(n-1)}_{\varphi(t)}(\varphi)$ at $\varphi(t)$. Given a proper face $F \in \mathcal{F}_B$, with $\{\ell_1, \ldots, \ell_i\} \subseteq \mathcal{L}$ linear functionals defining F, define the ideal*

$$J_F^{(n-1)} = \langle \ell_1(\varphi(t) - u) - \lambda, \ldots, \ell_i(\varphi(t) - u) - \lambda \rangle + \left\langle (i+1) \times (i+1) \text{ minors of } \begin{pmatrix} \eta_{n-1}(t) \\ \nabla \ell_1 \\ \vdots \\ \nabla \ell_i \end{pmatrix} \right\rangle$$

in $\mathbb{R}[t, u_1, \ldots, u_n, \lambda]$. Define $J^{(n-1)} := \bigcap_{F \in \mathcal{S}_B} J_F^{(n-1)}$, where $\mathcal{S}_B := \{F \in \mathcal{F}_B \mid \exists t \in \mathbb{A}^1, F \in$ $\mathrm{Type}^{(n-1)}(\varphi(t))\}$, and define the eliminated ideal $I^{(n-1)} := J^{(n-1)} \cap \mathbb{R}[u_1, \ldots, u_n]$. Then we have

$$\mathcal{PL}_{n-1}(\varphi) \subseteq \mathbb{V}(I^{(n-1)}).$$

Example 6.7. Note that for an $(n-1)$-regular algebraic morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$, it is generally not a straightforward computation to find the precise subset of faces $\mathcal{S}_B = \{F \in \mathcal{F}_B \mid \exists t \in \mathbb{A}^1, F \in \mathrm{Type}^{(n-1)}(\varphi(t))\}$. We can use the setup in Example 6.5 to compute these intersections and identify the corresponding faces.

Assuming the same setting as Example 6.5 for the twisted cubic curve and the 3-cube B. The intersections for the condition of higher order normal space occurs at the vertices of the cube and facets. By the central symmetry of the polytope B, antipodal vertices share the same eliminated ideal.

For the vertices $(1,1,1)$ and $(-1,-1,-1)$, the eliminated ideal is generated by the polynomial

$$u_1^3 - 3u_1^2u_2 + 3u_1u_2^2 - u_1u_2 + u_1u_3 - u_2^3 + 2u_2^2 - 3u_2u_3 + u_3^2$$

and for the vertices $(1,-1,1)$ and $(-1,1,-1)$, the eliminated ideal is generated by the polynomial,

$$u_1^3 + 3u_1^2u_2 + 3u_1u_2^2 - u_1u_2 - u_1u_3 + u_2^3 - 2u_2^2 - 3u_2u_3 - u_3^2.$$

Furthermore, intersection also happens at the 2-dimensional faces of the cube, given by vertices $\{(-1,-1,-1),(1,-1,-1),(-1,1,-1),(1,1,-1)\}$ (the $z=-1$ plane) and its opposite facet $\{(-1,-1,1),(1,-1,1),(-1,1,1),(1,1,1)\}$ (the $z=1$ plane). However the eliminated ideal returns the zero ideal $\langle 0 \rangle$. Therefore, like before in Example 6.5, the corresponding variety is the entire ambient space $\mathbb{A}^3$, since inner normal cone is full dimensional. If we plot the zero loci of these two polynomials obtained from the vertices, we get the geometric representation of the second-order polyhedral distance loci, as shown in the figure below:

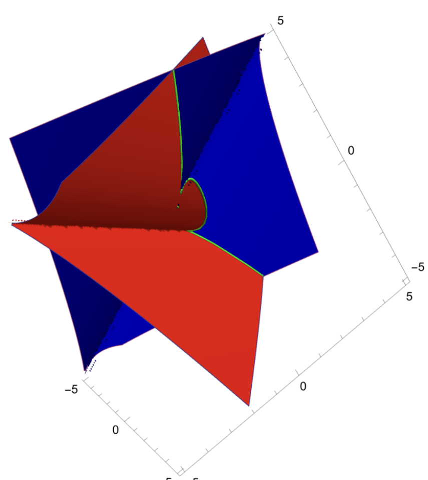

Figure 39: The second-order polyhedral distance loci for the twisted cubic curve.

Also, one can also see that Figure 39, is the Zariski closure of the Figure 35. $\triangle$

We now restrict the statement more. Recall that we initially defined

$$\mathcal{PL}_{n-1}(\varphi) = \bigcup_{t\in\mathbb{A}^1} \mathrm{VCone}(\varphi(t)).$$

We now subdivide $\mathbb{A}^1$ into intervals depending on the codimension of the face $F \in \mathrm{Type}^{(n-1)}(\varphi(t))$.

We first subdivide the set of proper faces $\mathcal{F}_B$ by their codimension. Let $i \in \{1,\dots,n\}$, then define $\mathcal{F}_B(i) = \{F \in \mathcal{F}_B \mid \operatorname{codim} F = i\}$, so that we have decomposition $\mathcal{F}_B = \bigcup_{i=1}^n \mathcal{F}_B(i)$. Similarly, we can decompose the set $\mathcal{S}_B = \{F \in \mathcal{F}_B \mid \exists t \in \mathbb{A}^1 : F \in \mathrm{Type}^{(n-1)}(\varphi(t))\}$ as $\mathcal{S}_B(i) = \{F \in \mathcal{S}_B \mid \operatorname{codim} F = i\}$ to have the decomposition $\mathcal{S}_B = \bigcup_{i=1}^n \mathcal{S}_B(i)$.

For each proper face $F \in \mathcal{F}_B$, we define its corresponding parameter interval as $A_F = \{t \in \mathbb{A}^1 \mid F \in \mathrm{Type}^{(n-1)}(\varphi(t))\}$, then the set $A(i) = \bigcup_{F\in\mathcal{S}_B(i)} A_F$ is a subset of $\mathbb{A}^1$ such that the parameter t is optimized at a face of codimension i. Therefore we can subdivide $\mathbb{A}^1$ into regions $A(1),\dots,A(n)$ based on the codimension of the faces. Using this notation, we can also decompose the polyhedral distance locus $\mathcal{PL}_{n-1}(\varphi)$ into its corresponding codimension components as

$$\mathcal{PL}_{n-1}(\varphi, i) = \bigcup_{t\in A(i)} \mathrm{VCone}^{(n-1)}(\varphi(t)),$$

which gives us the decomposition

$$\mathcal{PL}_{n-1}(\varphi) = \bigcup_{i=1}^{n} \mathcal{PL}_{n-1}(\varphi, i).$$

Now in order to not deal with the facets of the polyhedral ball B, we define truncated polyhedral distance loci to be

$$\widetilde{\mathcal{PL}}_{n-1}(\varphi) = \bigcup_{i=2}^{n} \mathcal{PL}_{n-1}(\varphi, i),$$

so that

$$\mathcal{PL}_{n-1}(\varphi) = \mathcal{PL}_{n-1}(\varphi, 1) \cup \widetilde{\mathcal{PL}}_{n-1}(\varphi).$$

Theorem 6.4. *Let* $\widetilde{\mathcal{S}}_B := \bigcup_{i=2}^n \mathcal{S}_B(i)$, *and* $\tilde{J}^{(n-1)} := \bigcap_{F\in\widetilde{\mathcal{S}}_B} J_F^{(n-1)}$, *and* $\widetilde{I}^{(n-1)} = \tilde{J}^{(n-1)} \cap \mathbb{R}[u_1,\dots,u_n]$, *then*

$$\widetilde{\mathcal{PL}}_{n-1}(\varphi) \subseteq \overline{\widetilde{\mathcal{PL}}_{n-1}(\varphi)} \subseteq \mathbb{V}(\widetilde{I}^{(n-1)})$$

where $\overline{\widetilde{\mathcal{PL}}_{n-1}(\varphi)}$ *is the Zariski closure of* $\widetilde{\mathcal{PL}}_{n-1}(\varphi)$.

Proof. First note that first inclusion $\widetilde{\mathcal{PL}}_{n-1}(\varphi) \subseteq \overline{\widetilde{\mathcal{PL}}_{n-1}(\varphi)}$ follows from the definition of Zariski closure. To show the second inclusion $\overline{\widetilde{\mathcal{PL}}_{n-1}(\varphi)} \subseteq \mathbb{V}(\widetilde{I}^{(n-1)})$, it is enough to show the $\widetilde{\mathcal{PL}}_{n-1}(\varphi) \subseteq \mathbb{V}(\widetilde{I}^{(n-1)})$, because $\mathbb{V}(\widetilde{I}^{(n-1)})$ is already closed in Zariski topology. Therefore it is enough to prove $\widetilde{\mathcal{PL}}_{n-1}(\varphi) \subseteq \mathbb{V}(\widetilde{I}^{(n-1)})$.

Let $u \in \widetilde{\mathcal{PL}_{n-1}}(\varphi)$, then by definition there is an $i \in \{2, \dots, n\}$ such that $u \in \mathcal{PL}_{n-1}(\varphi, i)$. Therefore, there is a $t \in A(i) \subseteq \mathbb{A}^1$ such that $u \in \mathrm{VCone}^{(n-1)}(p)$, where $p = \varphi(t)$. So there is a codimension i face F where $i \neq 1$, such that $F \in \mathrm{Type}^{(n-1)}(p)$, hence $F \in \widetilde{\mathcal{S}}_B$.

Now by similar argument in the proof of Theorem 6.2, there is $\lambda \in \mathbb{R}$ such that $(t, u, \lambda) \in \mathbb{V}(J_F^{(n-1)})$. Since $F \in \widetilde{\mathcal{S}}_B$, this implies $(t, u, \lambda) \in \mathbb{V}(\widetilde{J}_F^{(n-1)}) \subseteq \mathbb{V}(\widetilde{J}^{(n-1)})$. Eliminating the variables t and λ, implies $u \in \mathbb{V}(\widetilde{I}^{(n-1)})$, which completes the proof. ■

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